INCONSPICUOUSNESS AND OBFUSCATION: HOW LARGE SHAREHOLDERS DYNAMICALLY MANIPULATE OUTPUT AND INFORMATION FOR TRADING PURPOSES

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Abstract. I relate the theory of large shareholders in corporate governance to market microstructure theory. The large shareholder literature examines how a large shareholder trades off the advantage of being able to influence the decisions of the firm, while small shareholders free ride on the outcomes, against the extra risk entailed in large shareholdings. The market microstructure literature is concerned with the use of private information in pricing stocks.

The large shareholder can affect the underlying value of the firm not only in the conventional sense; he can also profit because this improves his ability to hide his private information from other informed traders and from market makers. In a static version of the model, the large shareholder increases the volatility of firm fundamentals, but only on the component of his private information that is unforecastable by the market: he obfuscates.

In dynamic models of stock markets with private information, informed traders, including the large shareholder, chop their orders in order to delay the impact of their information on prices so that they can be inconspicuous. Using Fourier transform methods to construct a continuous time dynamic version of the large shareholder model, I demonstrate that the large shareholder alters the fundamental autoregressive structure of the fundamental value of the firm because this improves his ability to hide his private information from other informed traders and from market makers, that is, to obfuscate.

1. Introduction

While CEO at Hewlett-Packard, Mark Hurd would have been privy to information about the firm that outside shareholders did not have, and he was able to act on that information by altering decisions about products, production, personnel and myriad other matters—and keep these decisions from the view of outside shareholders. Should HP, which is in large part a printer and personal computer company, enter a new market, say for enterprise software that would compete against SAP and Oracle? Hurd would have inside knowledge, and could approve that entry or not—and also trade on that information.

Hurd's trades would affect the market price of HP. Knowing that he is a large shareholder, Hurd would temper not only his trading but also his business decisions within HP.

By being a large shareholder in HP, Hurd would have taken on risks that would be potentially deleterious to him,[1] but his ability to monitor the firm and make

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[1]Assuming that he cannot diversify and balance his portfolio appropriately.

production decisions would enable him to appropriately minimize that risk. Smaller shareholders might benefit from his incentives and receive lower returns in exchange for the monitoring incentive—that is, they would free ride.

This is the perspective of the existing literature, with key papers by Admati, Pfleiderer and Zechner [1], Shleifer and Vishny [27], and De Marzo and Uroševic [16].

That the large shareholder will amplify the volatility of that part of the firm's fundamentals that he observes privately also seems intuitively reasonable: if you give a CEO options in the company stock, his or her incentive is to increase the volatility of the company stock price in order to increase the option price.

In this paper the large shareholder's behavior is stationary. The large shareholder is risk neutral, but risk drives his behavior because of his largeness in the market for the firm's shares, and he can affect the underlying value of the firm. In addition, he has private information about the underlying shock processes. Other informed large traders cannot affect firm value but do have private information as with the descendants of the standard Kyle [24] model such as Back, Cao and Willard [5], Holden and Subrahmanyam [21], and Foster and Viswanathan [17].

In these pure trading models, information is used by the market makers and by the informed rivals from current or past prices to impute the information of the informed traders. The informed traders know this, and attempt to trade on the part of their private signals that is unforecastable using public price information. As a result, total order flow looks like noise trade order flow; the informed traders hide behind the noise traders. The informed trades are thus *inconspicuous*.

An alternative way to interpret inconspicuousness is that the privately informed traders mask their trades so as to appear uninformed, and thus block other market participants from imputing the information. This inconspicuousness appears in other studies: Danilova [14], who coined the term, and Back and Baruch [4], each using very different technical frameworks, find this result.

As in other versions of the Kyle model, the expected profit of the informed trader is proportional to the product of the volatility of the fundamental value and the volatility of the noise trade. If the informed trader is also a large shareholder, he can affect the fundamental value of the stock by his actions, and so he can affect that profit by increasing the volatility of the value. That strategy emerges here.

In a static version of the model, the large shareholder increases the volatility of firm fundamentals in this way, but only on the unforecastable component of his private information: he obfuscates.

In reality, the shocks that impinge on firm value are dynamic. Inconspicuousness still emerges as a strategy: privately informed traders alter the dynamic (i.e. autoregressive) structure of their trades so that total order flow has the same dynamic structure as noise trade. But in addition, the large shareholder can affect the underlying value of the firm not only in the conventional static sense, he can affect the dynamic structure of the value, that is, its autoregressive structure. He can profit by altering this structure, because this improves his ability to hide his private information from other informed traders and from market makers. The large shareholder now not only increases volatility, he alters the serial correlation structure of the firm's fundamental value process.

Thus, not only do large shareholders increase the volatility of firm fundamentals, they make those fundamentals harder to glean from prices. This has implications that go beyond standard concerns about market liquidity.

1.0.1. *Outline.* The plan of the paper is as follows. First, I briefly review the literature on the volatility impact of managerial and shareholder incentives on firm value volatility. I then set out a static model with the ingredients of the basic Kyle [24] model: stocks, informed traders, market makers and noise traders, but with the additional feature that the first of the informed traders is also a large shareholder who can affect the fundamental value of the first stock via a costly action. The notation allows for the presence of multiple stocks and multiple informed traders, but for intuitive clarity I derive the first results in an example in which the large shareholder is the only informed trader.

Next, I develop a dynamic, continuous time version of the pure trading model. I use linear operator and control theory, known to economists as frequency domain methods, which allow a succinct characterization of how endogenous dynamics are affected by incentives and by equilibrium considerations; some details about the methods are in a technical appendix. I present a couple of basic results regarding the trading behavior of the informed traders, namely that in equilibrium they hide their trades, and I note how this relates to other modeling approaches. I then add the large shareholder to the dynamic model and demonstrate with these methods that his actions will alter the dynamic structure—that is, the time series or autoregressive structure—of the firm's fundamentals.

2. Literature

The conclusion that the large shareholder will amplify the volatility of that part of the firm's fundamentals that he observes privately seems intuitively reasonable: if you give a CEO options in the company stock, his incentive is to increase the volatility of the company stock price in order to increase the option price.[2] This has already been noted by the literature: recent references include Camara and Henderson [12], Goldman and Slezak [18], Peng and Roell [26], Goldstein and Guembel [19] and Bolton, Scheinkman and Xiong [10].

Peng and Roell empirically document the increase in shareholder litigation when managers are given option contracts as incentives. Their conclusion is that such contracts encourage the managers to focus on short term share prices.

Bolton, Scheinkman and Xiong develop a model in which the short term focus of the managers when they are given such option contracts is potentially desirable, because it enables them to increase the short term speculative component of the share price, benefiting current shareholders. That finding is mirrored in a sense here because the model here includes noise traders who are the source of profit for the informed traders. Both Peng and Roell and Bolton, Scheinkman and Xiong focus on managerial behavior, rather than large shareholder behavior.

Goldman and Slezak develop a model in which managers can exert agency-style effort to manipulate earnings so as to increase the value of their incentive pay. They conclude however that the manager's increased effort can actually increase

[2]Options can be a substantial part of pay: Meg Whitman, who recently took over Hewlett-Packard, was given compensation package consisting of a salary of $1 and options.

shareholder welfare because the efforts are in the right direction, that is, they improve firm value.

Goldstein and Guembel develop a model of price manipulation that is more focused on the production side. Firm managers observe prices in the market and make investment decisions based on those prices because they contain information that the manager might not be able to directly observe. Informed speculators, distinct from the managers, know this and manipulate prices to profit from the manager's response, rather than directly to their own private signals. This notion is reflected in the model here: the large shareholder observes market prices (including the prices of other stocks that might have correlated information) and acts on that information as well as his own.

Camara and Henderson analyze the effects of several types of incentive contracts, and the effects of penalties and risk aversion on manager behavior. Among other conclusions, they find that risk aversion limits the manager's incentive to increase firm volatility.

3. The static model

The static model is similar to the model of Bernhardt and Taub [8]. There are N informed traders,[3] each of whom receives a separate zero-mean Gaussian-distributed signal e_i of the value of the firm, and a firm whose stock is traded.[4] In that model, firm value was the sum of those exogenous signals. Here, one of the traders can affect the value of the firm via his actions. Firm value without action by the large shareholder is

$$v = \sum_{i=1}^{N} e_i \tag{1}$$

The e_i are potentially correlated, but to conserve notation in this introductory model I will focus on the uncorrelated case. The vector of fundamental inputs to value is e, with covariance matrix $Eee' = R$.

In addition to the informed traders, there are numerous uninformed traders or market makers, who attempt to impute the underlying value of the firm from order flow. Finally, there are noise traders whose trades are are unmotivated by explicit portfolio considerations and are modeled simply as noise.

3.0.2. *The large shareholder.* I assume the convention that the large shareholder is informed trader 1; the informed traders $2, \dots, N$ receive signals $e_2, \dots, e_N$ respectively but cannot influence those signals. The large shareholder chooses θ_1 to weight his private signal in order to alter fundamental value:

$$v = \sum_{i=1}^{N} e_i - \theta_1 e_1.$$

Because the fundamental e_1 is constructed to have a zero mean, it is immediately evident that the influence of θ_1 will ultimately be on the *variance* of the fundamental

[3] In the literature on the Kyle model, the informed traders can also be called insiders or speculators.

[4] It is straightforward to consider multiple stocks as was done in [8], but I will assume a single stock here.

e_1. This is a departure from previous approaches, which have taken the underlying value v as fixed.

The large shareholder's actions are costly. I assume that the cost takes a specific form: the large shareholder is penalized on the degree of amplification of his private information, that is, on $\theta_1 e_1$. One might think of this as the influence of an extremely boiled-down and abstract version of an incentive contract that is structured to constrain the large shareholder's use of his private information to prevent overexploitation of the company's resources. The part of the contract that is omitted here characterizes the beneficial impact of the large shareholder's actions on other shareholders; using a technical framework similar to the one here, this idea was pursued in [29].[5]

3.0.3. *Trades.* In the standard model such as that in Bernhardt, Seiler and Taub [9], an informed trader's trade x_i (whether or not he is the large shareholder) is a linear function of his private signal of value e_i, expressed as trading intensity coefficient b_i and of the net information in price, with intensity γ_i. Because the informed traders' trades influence the price, the informed trader's trade takes account of the presence of his own and other traders' reactions to their information:

$$x_i = b_i e_i + \gamma_i \left(\sum_{j=1}^{N} b_j e_j - b_1 \theta_1 e_1 + u \right) \tag{2}$$

where u is the noise trade.

3.0.4. *Pricing.* Market makers receive the aggregate orders from the informed traders and from the noise traders, but are unable to distinguish individual trades. Because they know the order flow includes a component from informed traders, they price stocks by using signal extraction. The linear structure of the model means that price is a linear function of order flow

$$p = \lambda \left(\sum_{i=1}^{N} x_i + u \right)$$

where λ is a projection coefficient that expresses the signal extraction that is being carried out.[6]

3.0.5. *The informed's trader's problem.* An informed trader i who is not a large shareholder solves

$$\max_{x_i} E \left[\left(\sum_{j=1}^{N} e_j - \theta_1 e_1 - \lambda \left(\sum_{j=1}^{N} x_j + u \right) \right) x_i \middle| e_i, p \right]$$

[5]In keeping with the terminology of DeMarzo and Uroševic, I will sometimes refer to the large shareholder as the agent. This is intuitively reasonable if the large shareholder is an owner-entrepreneur, but it is important to emphasize that in this model, the other informed traders are also large, in that they can move the price of the firm's shares by trading, but they cannot directly affect fundamentals. I will refer to these informed traders as outside investors or traders. This is a little different from DeMarzo and Uroševic's model, where the outside investors are atomistic.

[6]In fact, linearity of pricing is not immediate, but it can be shown that a linear equilibrium exists, and this paper focuses only on this possibility. The linearity of the pricing rule is developed in Back [3]. Further analysis of the uniqueness of linear equilibria is presented in Boulatov, Kyle and Livdan [11], and also Bernhardt and Taub [9].

This is a standard construction along the lines of the Kyle [24] approach: the informed trader chooses his order x_i taking account of the impact of his trade on the price. Also, the outside informed traders take the modification of the fundamental via the term $\theta_1 e_1$ as given, that is, they view $e_1 - \theta_1 e_1$ simply as another fundamental.

3.0.6. *The large shareholder's problem.* Trader 1 is the large shareholder, who not only has private information about one of the elements that affect fundamental value, but in addition he can affect that element.

Large shareholders maximize trading profits, choose an amplification factor θ_1, and also face a penalty on their amplification factor. Assuming the large shareholder is informed trader 1, they solve

$$\max_{x_1,\theta_1} E\left[\left(\sum_j e_j - \theta_1 e_1 - \lambda\left(\sum_j x_j + u\right)\right) x_1 - \frac{C}{2}(\theta_1 e_1)^2 \middle| e_1, p\right] \tag{3}$$

where C is a constant penalty on the magnitude of the large shareholder's realized amplification $\theta_1 e_1$.

The solution of the outside shareholder informed trader's problem is a straightforward exercise: simply repeat the analysis of the Bernhardt and Taub [8] model, but with the first element of the value modified by the actions of the large shareholder. By restricting strategies to be linear in the elements of their information, the expectation can be carried through the objective prior to, rather than subsequent to, the calculation of the optimum. It is established in Bernhardt and Taub [8] that this reversal does not alter the outcome of the optimization. The choice variables in the optimization then change, from choosing order flow x_i directly, to choosing it indirectly by choosing the coefficients of the information vector, namely b_i and γ_i—the *trading intensities.*

Expressing the trading strategies explicitly in terms of b_i and γ_i as in (2), then carrying through the expectation in this way and with the assumption that the private signals are uncorrelated, the optimization problems of the informed traders can then be stated as:

$$\begin{aligned}\max_{\{b_i,\gamma_i\}} \Bigg\{ & \left((1-\theta_1) - b_1\left(1+\sum\gamma_j\right)\lambda\right)\gamma_i b_1 \sigma_1^2 \\ & + \sum_{2,j\neq i}^{N}\left(1 - b_j\left(1+\sum\gamma_j\right)\lambda\right)\gamma_i b_j \sigma_j^2 \\ & + \left(1 - b_i\left(1+\sum\gamma_j\right)\lambda\right)(1+\gamma_i) b_i \sigma_i^2 \\ & - \left(1+\sum\gamma_j\right)\lambda\gamma_i\sigma_u^2 \Bigg\}\end{aligned} \tag{4}$$

for the outside informed trader i, and

$$\max_{\{b_1,\gamma_1,\theta_1\}} \left\{ \left((1-\theta_1) - b_1\left(1+\sum\gamma_j\right)\lambda\right)(1+\gamma_1)b_1\sigma_1^2 + \sum_{j=2}^{N}\left(1-b_j\left(1+\sum\gamma_j\right)\lambda\right)\gamma_1 b_j\sigma_j^2 - \left(1+\sum\gamma_j\right)\lambda\gamma_1\sigma_u^2 - \frac{C}{2}\theta_1^2\sigma_1^2 \right\} \tag{5}$$

for the large shareholder.

3.1. **The large shareholder's first order conditions.** The first-order condition for the private-signal intensity b_i will now have some extra terms due to the amplification factor, and there is an entirely new first-order condition for θ_1.

The first-order condition for b_1 can be written as follows:

$$\left[\left((1-\theta_1) - b_1\left(1+\sum\gamma_j\right)\lambda\right)(1+\gamma_1) - b_1(1+\gamma_1)\lambda\left(1+\sum\gamma_j\right)\right]\sigma_1^2 = 0$$

Defining

$$H \equiv 1+\sum\gamma_j \tag{6}$$

the solution for b_1 is

$$b_1 = \frac{1}{2\lambda H}(1-\theta_1). \tag{7}$$

Thus, the large shareholder's trading intensity on private information is modified by his alteration of the variance of the fundamental.

3.1.1. *The first order condition for γ_1.* The first order condition for γ_1 is the same as in the pure trading model, but with the value process replaced by the large-shareholder-modified process.

$$\left((1-\theta_1) - b_1\left(1+\sum\gamma_j\right)\lambda\right)b_1\sigma_1^2 + \sum_{j=2}^{N}\left(1-b_j\left(1+\sum\gamma_j\right)\lambda\right)b_j\sigma_j^2 - \left(1+\sum\gamma_j\right)\lambda\sigma_u^2 - (b_1\lambda)(1+\gamma_1)b_1\sigma_1^2 + \sum_{j=2}^{N}(1-b_j\lambda)\gamma_1 b_j\sigma_j^2 - \lambda\gamma_1\sigma_u^2 = 0 \tag{8}$$

After substituting the solution for the b_j and the other γ_j this reduces to

$$\frac{1}{2}(1-\theta_1)b_1\sigma_1^2 + \frac{1}{2}\sum_{j=2}^{N}b_j\sigma_j^2 - \left(1+\sum\gamma_j\right)\lambda\sigma_u^2 - (b_1\lambda)(1+\gamma_1)b_1\sigma_1^2 + \sum_{j=2}^{N}(1-b_j\lambda)\gamma_1 b_j\sigma_j^2 - \lambda\gamma_1\sigma_u^2 = 0 \tag{9}$$

Further simplification is possible; I defer this to the example for $N=1$ below; the more general case is treated in [6].

The coefficient γ_1 has the following interpretation: it is the (negative of the) projection coefficient of the informed trader's trade on public information onto publicly available information, namely price; in turn, price is informationally equivalent to total order flow.[7] Because γ_1 is negative (see the example below), the factor $1+\gamma_1$ is the coefficient of the forecast error of that projection.

3.1.2. *The first order condition for θ_1.* The first-order condition for θ_1 is

$$-(1+\gamma_1)b_1\sigma_1^2 - C\theta_1\sigma_1^2 = 0.$$

The solution is

$$\theta_1 = -\frac{(1+\gamma_1)b_1}{C}. \tag{10}$$

Thus, θ_1 is proportional to the market maker's forecast error coefficient $(1+\gamma)$ on the traded part of the large shareholder's private signal, b_1; this is the same quantity on which the informed trader's orders are based.

3.2. **An example.** I next consider an example in which there is a single informed trader ($N=1$). This section mostly applies the model in [6], with $N=1$ and $\rho=0$. The formulas for the equilibrium quantities on page 7 of [6] (changing to the notation here, so that b is the coefficient on the private signal) are:

$$b = \frac{1}{2\lambda H} \quad \gamma = -\frac{b^2\sigma_1^2}{b^2\sigma_1^2+\sigma_u^2} \quad \lambda = \frac{1}{H}\frac{b\sigma_1^2}{b^2\sigma_1^2+\sigma_u^2} \tag{11}$$

where H is as defined in (6), and with the second equality following because $N=1$. Solving the three equations yields the equilibrium quantities

$$b = \frac{\sigma_u}{\sigma_1} \quad \gamma = -\frac{1}{2} \quad \lambda = \frac{\sigma_1}{\sigma_u} \quad H = \frac{1}{2} \tag{12}$$

Now we can solve for profit and for the forecast error variance. Using formula (12) from [6], profit is

$$\pi = \lambda H\sigma_u^2 = \frac{1}{2}\frac{\sigma_1}{\sigma_u}\sigma_u^2 = \frac{\sigma_1\sigma_u}{2}. \tag{13}$$

The forecast error variance is

$$\begin{aligned} E\left[(e-\lambda(x+u))^2\right] &= E\left[(e-\lambda(Hbe+Hu))^2\right] = (1-\lambda Hb)^2\sigma_1^2 + (\lambda H)^2\sigma_u^2 \\ &= (1-\frac{\sigma_1}{\sigma_u}\frac{1}{2}\frac{\sigma_u}{\sigma_1})^2\sigma_1^2 + (\frac{\sigma_1}{\sigma_u}\frac{1}{2})^2\sigma_u^2 = \frac{\sigma_1^2}{4}+\frac{\sigma_1^2}{4} = \frac{\sigma_1^2}{2} \end{aligned} \tag{14}$$

where we have used the formula for total order flow from equation (12) in [6].

Observe that we have a result similar to that in [7]: noise trade is irrelevant for the information content of price.

3.3. **Obfuscation: The effect of the large shareholder in the example.** Now we can analyze the effect of the large shareholder construct. First observe that because the large shareholder modifies the fundamental value of the firm, from the market maker's perspective that fundamental value is $(1-\theta_1)e_1$ with variance $(1-\theta_1)^2\sigma_1^2$. Thus, define

$$\tilde{e} \equiv (1-\theta_1)e_1 \qquad \tilde{\sigma}_1^2 \equiv (1-\theta_1)\sigma_1 \tag{15}$$

Now the model can be solved for b and λ using $\tilde{e}$ as the fundamental. From the solution in equation (10),

$$\theta_1 = -\frac{\frac{\sigma_u}{\tilde{\sigma}_1}}{2C}$$

[7]These assertions are elaborated in Bernhardt, Seiler and Taub [9] and in Seiler and Taub [28].

Proposition 1. θ_1 *is negative.*

Proof: The full solution for θ_1 is complicated because $\tilde{\sigma}$ is itself a function of θ_1. We have

$$\theta_1 = -\frac{\frac{\sigma_u}{(1-\theta_1)\sigma_1}}{2C}$$

yielding a quadratic in θ_1, with one positive and one negative solution. For large values of C, these solutions approach $+1$ and -1 respectively; clearly the positive solution, which would reduce the variance of the fundamental and incur a large penalty, is suboptimal. Thus, it is appropriate to view the negative solution as the economically appropriate solution. □

Because θ_1 is negative, the modified fundamental $e_1 - \theta_1 e_1$ is in fact an amplification of the fundamental variance. But in addition, note that this term too is proportional to the forecast error coefficient $1 + \gamma_1$. Thus, the amplification is only on the unforecastable part of the fundamental, limited only by the penalty C. I summarize the result as follows.

Proposition 2. *The large shareholder amplifies the unforecastable part of his component of the fundamental.*

Proof: Using a projection algebra argument, Bernhardt and Taub ([6], p. 11) demonstrate that the order flow x_i is comprised of a linear function of the market maker's forecast error of the informed trader i's private signal e_i. For the large shareholder, this translates to

$$x_1 = \frac{1+\gamma_i}{\lambda} E\left[(1-\theta_1)e_1 + \sum_2^n e_j \middle| \left((1-\theta_1)e_1 - E\left[(1-\theta_1)e_1 \middle| \sum_{j=1}^N x_j + u\right]\right)\right]$$

where it should be noted that the inner expectation is conditioned on total order flow $\sum_{j=1}^N x_j + u$. Factoring $1 - \theta_1$ out of this expression, we have

$$x_1 = (1-\theta_1)\frac{1+\gamma_i}{\tilde{\lambda}} E\left[(1-\theta_1)e_1 + \sum_2^n e_j \middle| \left(e_1 - E\left[e_1 \middle| \sum_{j=1}^N x_j + u\right]\right)\right] \tag{16}$$

where $\tilde{\lambda}$ is the pricing coefficient when there is a large shareholder. By Proposition 1, θ_1 is negative, and so the key effect of the large shareholder is to not only amplify the private signal, but to amplify the *unforecastable part* of the signal. □

Corollary 3. *The large shareholder's amplification of the unforecastable part of his component of the fundamental increases his profit.*

Proof: Recall that profit is proportional to the product of the volatility of the value and the noise trade $\sigma_1\sigma_u$. With crude notation, the large shareholder's gross profit is proportional to

$$(1-\theta_1)\sigma_1\sigma_u$$

although cost must be subtracted as well. Again, because $1 - \theta_1$ exceeds unity, the large shareholder's action serves to *amplify* the fundamental value e_1 and thus his profit. □

Thus, the large shareholder does in fact conform to intuition: he will effectively increase the volatility of the firm's value, and he takes advantage of this excess volatility in the dimension in which he receives signals in his trading. Thus, the large shareholder has a larger profit than a correspondingly informed outsider; these profits come at the expense of the noise traders.

The increase in the effective fundamental variance increases the pricing coefficient λ, as would be expected: the signal to noise ratio has been increased. The volatility of price also increases:

$$E\left[(\lambda H(b(1-\theta_1)e+u))^2\right]=\frac{\tilde{\sigma}_1^2}{2}$$

In addition, the forecast error variance also increases. This is not obvious a priori, because the higher variance of the fundamental σ_1^2 raises the signal to noise ratio in order flow. This would be expected to increase λ—which it does—and thus reduce the overall forecast error variance. However, the way the signal is amplified is via the forecast error of the signal, and intuitively this should not improve the signal component, and this is the result.

Proposition 4. *The large shareholder amplifies the market maker's forecast error variance.*

Proof: Specifically, we can note that $\tilde{\lambda}$ is

$$\tilde{\lambda}=\frac{\tilde{\sigma}_1}{\sigma_u}=\frac{(1-\theta_1)\sigma_1}{\sigma_u}$$

Recalling the formula for the large shareholder's order flow x_i from (16),

$$\begin{aligned}
x_1&=(1-\theta_1)\frac{1+\gamma_i}{\tilde{\lambda}}E\left[(1-\theta_1)e_1+\sum_2^n e_j\middle|\left(e_1-E\left[e_1\middle|\sum_{j=1}^N x_j+u\right]\right)\right]\\
&=(1-\theta_1)^2\frac{1+\gamma_i}{\frac{(1-\theta_1)\sigma_1}{\sigma_u}}E\left[e_1+\sum_2^n e_j\middle|\left(e_1-E\left[e_1\middle|\sum_{j=1}^N x_j+u\right]\right)\right]\\
&=(1-\theta_1)\frac{1+\gamma_i}{\frac{\sigma_1}{\sigma_u}}E\left[e_1+\sum_2^n e_j\middle|\left(e_1-E\left[e_1\middle|\sum_{j=1}^N x_j+u\right]\right)\right]
\end{aligned}$$

leaving intact the assertion that the large shareholder amplifies the market maker's forecast error. □

The fact that the large shareholder amplifies the *non-forecastable* part of his order flow—obfuscation—suggests that in a dynamic setting the large shareholder might want to alter the time series structure of the fundamental. This conjecture is true and in the next section I set the groundwork for demonstrating this. The method used in the static model—restricting actions to be linear functions of the information realizations, and then solving for the coefficients of those linear polices—works in the dynamic setting as well, but requires functional analysis tools.

4. Adding dynamics

I will now set out a dynamic version of the model using a continuous time approach. The approach and structure is much like that of [28], except that it is set in continuous time, and some notation and methods need to be translated. After noting some informational characteristics of the equilibrium, I then expand the model to examine the effects of the large shareholder on firm value and price dynamics.

In Bernhardt and Taub [8], the cross-asset covariance structure of prices is driven strictly by the covariance structure of the underlying assets, but not of the noise trade, while cross-asset order flow is driven strictly by the covariance structure of the noise trade, but not of the assets. As mentioned in the introduction this is because the informed traders hide behind the noise trade. Correspondingly, the market makers structure the price process so that no arbitrage can be done against them by the informed traders that would be enabled if the price differed in structure from the value.

These results extend to the dynamic setting: the vector autoregressive structure of the order flow vector process mimics the autoregressive structure of the noise trade process, and the price vector process mimics the autoregressive structure of the fundamental firm value process. The order flow process will depend on the covariance structure of the noise trade process, and not on the covariance structure for the value process. The opposite is true for the price process: the price process will depend on the covariance structure of the value process, but not on the covariance structure of the noise trade process.

I use the continuous-time analogue of Bernhardt, Seiler and Taub [9] and Seiler and Taub [28] in order to carry out the dynamic analysis. The main tools are the Laplace and Fourier transforms and the continuous-time analogue of the Wiener-Hopf equation. These tools are described in sections 6.A (pp. 216-220), 7.1-7.2 (pp. 221-228), and 7.A (262-264) of Kailath, Sayed and Hassibi [23]. An additional reference is Hansen and Sargent [20].

In the standard market microstructure literature, the noise trade is assumed to be a Brownian process, while there is a single realization of the underlying asset value, as in the original Kyle model [24], or another Brownian process as in Danilova's recent model [14]. Thus, both noise trade and information are highly persistent. By contrast, in this paper while I maintain the assumption of a persistent value process, I assume that the noise trade is serially uncorrelated. At least at a logical level this is a more satisfying assumption, as it precludes persistent errors on the part of noise traders. But it also serves to highlight the starkly different dynamic structure of order flow and prices, differences that highlight the economic forces driving those processes.

In the next section I develop the technical elements of the dynamic model without the large shareholder, building on previous work. The conclusions regarding inconspicuousness are easily established in this framework. In subsequent sections I add the large shareholder and develop the main result.

5. The dynamic model without the large shareholder

In order to set the stage for the large shareholder analysis, I first set out the model without the large shareholder. Some basic propositions can be stated, some of which reiterate and highlight the findings in [9] and [28]. The first step is to write down the objectives of the informed traders and of the market makers, and to derive the structure of the aggregate information formula.

5.1. The setting and the informed shareholders' objectives.

In this paper I begin with the assumption that the total cost of shares $X(t)$ held at each time t is proportional to the cost of acquiring them at each instant, $\int_0^t P(s)x(s)ds$, where $x(s)ds$ is the incremental shares acquired.

In the standard setup of the Kyle model, the underlying value is fixed, and is revealed at the terminal time T. There are two differences here: first, the horizon is infinite, and so the underlying value is effectively never revealed. Second, the value fluctuates stochastically, and meaning must be given to this fluctuation. The interpretation will be that at each moment there is a possibility that the firm will be bought, merged or terminate, with some probability that is independent of past or current states and associated hazard rate δ; I will refer to this as *conversion.* Should the conversion occur, the payoff for an informed trader is

$$v(t)\int_0^t x(s)ds$$

where $v(t)$ is the time-t realization of the stochastically evolving fundamental value, and this happens with probability $\delta e^{-\delta t}$. The probability-weighted expected payoff at time t is then

$$E\left[\delta e^{-\delta t}v(t)\int_0^t x(s)ds\middle|\omega(0)\right]$$

where $\omega(t)$ is the trader's information at t. Thus, the expected profit over all dates of conversion is

$$E\left[\int_0^\infty \delta e^{-\delta t}v(t)\int_0^t x(s)\,ds\,dt\middle|\omega(0)\right]$$

By changing the order of integration we can write the inner terms as

$$\int_0^\infty x(t)\int_t^\infty \delta e^{-\delta s}v(s)\,ds\,dt \tag{17}$$

Defining the probability-discounted value of the asset at any moment as

$$V(t)\equiv\int_t^\infty \delta e^{-\delta t}v(s)\,ds$$

we can write equation (17) as

$$E\left[\int_0^\infty x(t)V(t)dt\middle|\omega(0)\right]. \tag{18}$$

This then justifies writing discounted expected profit as

$$E\left[\int_0^\infty (V(t)-P(t))x(t)dt\middle|\omega(0)\right]. \tag{19}$$

5.1.1. *Value and noise trade process specifics.* Let $e_i(t)$ be the private information process for trader i; for the moment I assume that these are zero-mean white noise processes[8] that are uncorrelated across traders, and it remains the case that the firm value is the sum of these signals. Now however, the signals and consequently firm value are stochastic processes. The value process is a filtered version of these fundamental signal processes:

$$v(t) = \int_0^\infty \phi(\tau) \sum_{i=1}^{N} e_i(t-\tau) d\tau \tag{20}$$

Similarly, the noise trade process can be a filtered version of a fundamental white noise process $n(t)$:

$$u(t) = \int_0^\infty \nu(\tau) n(t-\tau) d\tau.$$

Both the value and noise processes are characterized by the filters ϕ and ν. For the purposes of characterizing the model it will be assumed when necessary that the value and noise trade processes are Ornstein-Uhlebeck processes, that is, analogues of autoregressive processes in discrete time settings. The filters are then in exponential form:

$$\phi(\tau) = e^{-\rho\tau} \qquad \nu(\tau) = e^{-\eta\tau}.$$

These processes become Brownian motions at $\rho = 0$ and $\eta = 0$, and white noise processes at the other extreme, $\rho = \infty$ and $\eta = \infty$. Economic intuition suggests that the value processes should be highly predictable; similarly intuition suggests that noise trade should not be persistent. To keep the model tractable I will assume that noise trade is white noise ($\eta = \infty$), but that the value process can have any degree of persistence, characterized by $0 \le \rho < \infty$.

Using the continuous time transform as described in Kailath, Sayed and Hassibi [23] p. 217, and also in Appendix A, the s-transforms of the filters for the value and noise trade processes are then $\Phi = \frac{1}{s+\rho}$ and the identity matrix I respectively.

5.2. **Informed traders' order flow.** Also, let $\Omega(t)$ be the public information process, which is going to be equivalent to the information in price. The informed trader's trading strategy process is a linear filtering of the histories of these processes:

$$x_i(t) = \int_{\tau=0}^\infty (b_{i,\omega}(\tau) e_i(t-\tau) + b_{i,\Omega}(\tau)\Omega(t-\tau)) d\tau.$$

In addition, noise traders exogenously submit an order flow process $u(t)$. Adding up the informed and noise trade yields total order flow:

$$\sum_{i=1}^{N} x_i(t) + u(t) = \sum_{i=1}^{N} \left(\int_{\tau=0}^\infty (b_{i,\omega}(\tau) e_i(t-\tau) + b_{i,\Omega}(\tau)\Omega(t-\tau)) d\tau \right) + u(t). \tag{21}$$

[8]See [23] p. 218 or [20] p. 209.

Defining the left hand side as $\Omega(t)$, we have

$$\begin{aligned}
(22) \quad \Omega(t) &= \sum_{i=1}^{N} \left(\int_{\tau=0}^{\infty} (b_{i,\omega}(\tau) e_i(t-\tau) + b_{i,\Omega}(\tau)\Omega(t-\tau)) d\tau \right) + u(t) \\
&= \sum_{i=1}^{N} \int_{\tau=0}^{\infty} b_{i,\omega}(\tau) e_i(t-\tau) d\tau + \sum_{i=1}^{N} \int_{\tau=0}^{\infty} b_{i,\Omega}(\tau)\Omega(t-\tau) d\tau + u(t).
\end{aligned}$$

This is a convolution as described in Kailath, Sayed and Hassibi [23] p. 217, but the integration is one-sided. However, if it is known in advance that the functions $b_{i,\omega}$ and $b_{i,\Omega}$ are one-sided, the integral can be converted to two-sided form and obtain the Laplace transform of the equation, following Kailath, Sayed and Hassibi's convention of using capital letters for the Laplace transform:

$$O(s) = \sum_{i=1}^{N} B_i(s) E_i(s) + \sum_{i=1}^{N} B_{i,\Omega}(s) O(s) + U(s).$$

where $B_i(s)$ is the transform of $b_{i,\omega}$. (See pp. 216-217 of [23].) The convolutions have been converted into products as a result of the transform. Now it is possible to solve for $O(s)$. This yields

$$(23) \qquad O(s) = \left(1 - \sum_{i=1}^{N} B_{i,\Omega}(s) \right)^{-1} \left(\sum_{i=1}^{N} B_i(s) E_i(s) + U(s) \right)$$

The procedure here was to convert to the Laplace transform and then solve—this is different from the order in discrete time, where the time domain problem was solved directly once the lag operator form was used.

The solution approach used in [9] can now be followed: define γ_i as the filter characterized by the transform

$$(24) \qquad \Gamma_j(s) = \left(1 - \sum_{i=1}^{N} B_{i,\Omega}(s) \right)^{-1} B_{j,\Omega}(s)$$

and then substituting from equation (22) into the order flow equation (21) and using (23) and (24), the order flow process becomes

$$\begin{aligned}
(25) \quad x_i(t) = &\int_{\tau=0}^{\infty} b_{i,\omega}(\tau) e_i(t-\tau) d\tau \\
&+ \int_{\tau=0}^{\infty} \gamma_i(\tau) \left(\sum_{j=1}^{N} \int_{\sigma=0}^{\infty} b_{j,\omega}(\sigma) e_j(t-\tau-\sigma) d\sigma d\tau + u(t-\tau) \right) d\tau
\end{aligned}$$

The bracketed term can then be viewed as the public information process inherent in the price process.

Assuming linear pricing, the price process is determined by a linear filter λ, with transform Λ, applied to total order flow:[9]

(26)

$$p(t) = \int_0^\infty \lambda(\tau) \left[\int_{\sigma=0}^\infty \int_{\nu=0}^\infty \sum_{j=1}^N b_{j\omega}(\sigma) \left(1 + \sum_{k=1}^N \gamma_k(\nu)\right) e_j(t-\nu-\sigma-\tau) d\nu d\sigma \right. \\ \left. + \sum_{k=1}^N \int_{\nu=0}^\infty \gamma_k(\nu) u(t-\nu-\tau) d\nu \right] d\tau$$

These ingredients will now be combined to form the objective for the informed traders.

5.3. **The informed trader's objective.** Expressing the informed trader's actions in terms of the filters expressing the value process in equation (20), the price process in (26), and the order flow process from (25) which includes the public information process, we can write the time-domain objective (19) for informed trader i as

(27)

$$\max_{\{b_{i\omega}, \gamma_i\}} E \int_0^\infty e^{-rt} \left(\int_{\tau=0}^\infty \phi(\tau) \sum_{j=1}^N e_i(t-\tau) d\tau \right. \\ - \int_{\tau=0}^\infty \lambda(\tau) \left[\int_{\sigma=0}^\infty \int_{\nu=0}^\infty \sum_{j=1}^N b_{j\omega}(\sigma) \left(1 + \sum_{k=1}^N \gamma_k(\nu)\right) e_j(t-\nu-\sigma-\tau) d\nu d\sigma \right. \\ \left. \left. + \sum_{k=1}^N \int_{\nu=0}^\infty \gamma_k(\nu) u(t-\nu-\tau) d\nu \right] d\tau \right) \\ \times \left(\int_{\tau=0}^\infty b_{i,\omega}(\tau) e_i(t-\tau) d\tau + \int_{\tau=0}^\infty \gamma_i(\tau) \left(\sum_{j=1}^N \int_{\sigma=0}^\infty b_{j,\omega}(\sigma) e_j(t-\tau-\sigma) d\sigma + u(t-\tau) \right) d\tau \right) dt$$

This objective is nontrivial, because the choice of the optimal action each period is conditioned on information, which includes the history of endogenous actions. As in the static model, the solution is more straightforward if the objective is first converted to frequency domain form, with the choice variables converted from time-domain period-by-period actions to the choice of optimal *filters* in the frequency domain.[10] I next develop the recipe for converting the objective to frequency domain form.

6. Optimizing in the frequency domain

Whiteman [32] constructed a discrete time model and then converted the objective itself into z-transform form. The optimization was then over linear operators or filters that were found via a variational derivative of the transformed objective.[11] This was achieved by imposing the constraint that the controls must be a linear

[9]Again, it should be noted that the linearity of the price process here is an assumption; as previously noted the validity of this assumption for existence has been explored in previous papers such as [9]; the necessity of linear pricing in the standard Kyle model was established in [3].

[10]The equivalence of these formulations was explored in [9].

[11]An earlier instance of the method is in Davenport and Root [15].

filter of the information, and taking the expectation of the objective prior to optimizing over those filters; this is the extension of the similar operation that was carried out in going from equation (3) to equation (4) in the static model. However, it is essential to reduce the covariance function of the fundamental processes—the white noise fundamentals—to a scalar covariance matrix. In continuous time, the equivalent operation is to make the fundamental covariance function $R_x(t)$ a Dirac δ-function.

If the fundamental processes are serially uncorrelated, as is the case here by the assumption that the fundamental processes are white noise, then the expectation of an objective like (27) leaves an integral in which the integrand consists of products of functions. Fourier transforming these objects then yields a convolution in the frequency domain, and the variational derivative of these convolutions can then be calculated. Proceeding in this way with abstract functions f and g,

$$\int_0^{\infty} e^{-rt} f(t)g(t)dt = \int_{a-i\infty}^{a+i\infty} F(s)G^*(r-s^*)ds$$

where the notation G^* signifies the complex conjugate transpose of G, $G^*(r-s^*)$.[12] the $r-s^*$ term captures discounting, and where the integration is along a strip parallel to the imaginary axis in which $\mathrm{Re}(s) = a$, where the functions F and G are analytic in the right half plane—that is, F and G have no poles or singularities in the region $\mathrm{Re}(s) > -r$, and with a small enough to avoid poles and thus yield convergence, that is, $a < r$.[13] There are two parts to the integrand: the "causal" part $F(s)$ and the "anti-causal part" $G^*(r-s^*)$, reflecting the inner product that is expressed in the objective.

6.1. **The informed trader's problem.** Applying this method to the informed trader's problem, the transformed objective will be a function of the value process filter $V(s)$, the public information process filter $O(s)$, the pricing filter $\Lambda(s)$, and the informed trader's order flow $X_i(s)$. The transform of the objective (27) is then

$$\max \int_{a-i\infty}^{a+i\infty} (V(s) - \Lambda(s)O(s))X_i^*(r-s^*)Rds \tag{28}$$

where the causal and anti-causal parts reflect the inner product that is expressed in the objective, and where R is the covariance matrix function of the Dirac-δ fundamentals $e_i(t)$ and $u(t)$.[14] To keep the model tractable, I will assume as in [9] and [28] that the noise trade process is uncorrelated with the fundamental value and signal processes, so the covariance function R is block diagonal:

$$R = \begin{pmatrix} R_e & 0 \\ 0 & R_u \end{pmatrix}. \tag{29}$$

The internal pieces of X_i and V, Λ, and O can now be broken out. The causal and anti-causal pieces ($V - \Lambda O$ and X_i^* respectively) are such that the Fourier transform of a sum is the sum of the Fourier transforms. In discrete time the convolution of functions of the lag operator translates into multiplication of functions in the z-domain. This also holds true in the continuous time setting: let

[12] For a concrete example in which this integration is calculated, see the proof of Lemma 9 in Appendix B.

[13] Notice that as in the discrete time case discounting weakens the constraints on poles.

[14] Again, see [23] p. 218 or [20] p. 209.

$g(t) = \int_0^\infty h(\tau)m(t-\tau)d\tau$, and consider the Fourier transform of $\int_0^\infty f(\sigma)g(t-\sigma)d\sigma$. Then it is immediate that

$$F(s)G(s) = F(s)(H(s)M(s)).$$

With this result in hand we can write the Fourier transformed objective (28) with the explicit decomposition of the price process. Also, $B_i(s)$ is the Fourier transform of the filter $b_{i\omega}(t)$; $\Gamma_i(s)$ is the Fourier transform of $\gamma_i(t)$, and H is the Fourier transform of $1 + \sum_{j=1}^N \gamma_j(t)$, so that

$$H(s) = 1 + \sum_{i=1}^N \Gamma_i(s). \tag{30}$$

analogously with equation (6) in the static model. With these ingredients trader i's transformed order flow filters are a vector of transforms

$$\begin{pmatrix} B_1\Gamma_1 & \dots & B_i(1+\Gamma_i) & \dots & B_N\Gamma_N \end{pmatrix} \tag{31}$$

with each element corresponding to the filters operating on the separate fundamental processes $e_j(t)$; the extra term in the ith element adds i's direct operation on his own signal. Similarly, the price process transform consists of the elements

$$\begin{pmatrix} \Lambda B_1 H & \dots & \Lambda B_N H & \Lambda H \end{pmatrix}$$

operating on the individual fundamental processes $e_j(t)$ and the noise trade process, and where the total order flow by all agents is captured by adding up the individual transforms in (31) and using the compact notation in (30). Finally, Φ_i is the Fourier transform of the signal process that informed trader i sees, and Φ is the vector of these signals, which sum to the value process of the stock.

Combining these ingredients yields the s-transform of the objective (27),

(32)

$$\max_{\{B_i,\Gamma_i\}} -\int_{a-i\infty}^{a+i\infty} \operatorname{tr}\left\{ \begin{pmatrix} \Phi - B_1 H\Lambda \\ \vdots \\ \Phi - B_N H\Lambda \\ -H\Lambda \end{pmatrix} \begin{pmatrix} \Gamma_i^* B_1^* & \cdots & (1+\Gamma_i^*)B_i^* & \cdots & \Gamma_i^* B_N^* & \Gamma_i^* \end{pmatrix} R \right\} ds$$

which parallels equation (7) of [9].[15]

6.2. **The informed trader's first-order condition for B_i.** Following the steps in [9], the first-order conditions of the s-transformed objectives can be stated. First, the notation

$$\mathcal{A}^*$$

denotes an arbitrary function is the s-domain that is anti-causal, that is, $A^*(s) = 0$, for s in the right half-plane.

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[15]The notation extends to the more general situation in which there are multiple assets. In that case, the value process $v(t)$ is a M-vector defined by

$$v(t) = \int_0^\infty \phi(\tau)e(t-\tau)$$

with the fundamental N-dimensional vector process $e(t-\tau)$, and $\phi(\tau)$ a matrix filter with M rows, one for each asset. In that case the s-transform Φ of ϕ is also a matrix. In that case, each informed trader i now submits a vector of trades, with $b_i(\tau)$ a $M \times 1$ vector that takes account of the influence of his signal on each of the M stocks.

Focusing on the B_i first-order condition, and assuming no cross-correlation of information, the first-order condition for B_i is

$$\Big[(\Phi - B_i H \Lambda)\,(1+\Gamma_i^*) - B_i(1+\Gamma_i)\Lambda^* H^* \Big] \sigma_{ie}^2 = \mathcal{A}^*.$$

In the uncorrelated case the elements are all scalars and will commute; the solution methods for continuous-time Wiener-Hopf equations outlined in Kailath, Sayed and Hassibi section 7.A can now be used. Gather terms to restate the equation as

$$B_i \Big[\Lambda H (1+\Gamma_i^*) + (1+\Gamma_i) H^* \Lambda^* \Big] \sigma_{ie}^2 = \Phi(1+\Gamma_i^*)\sigma_{ie}^2 + \mathcal{A}^*$$

Now propose a factorization

$$G_i G_i^* \equiv \Lambda H (1+\Gamma_i^*) + (1+\Gamma_i) H^* \Lambda^*$$

where by standard results G_i is analytic and invertible. Then the solution is

$$B_i = \Big\{ \Phi(1+\Gamma_i^*) G_i^{*-1} \Big\}_+ G_i^{-1} \tag{33}$$

where the projection operator $\{\cdot\}_+$ is defined by

$$\{F(s)\}_+ = 0, \qquad \mathrm{Re}(s) \leq 0$$

Some interpretation of (33) is possible. The solution for B_i is the s-transform analogue of a projection coefficient. There are two elements in the "numerator" or covariance part of this projection coefficient: Φ, the filter characterizing the informed trader's information, and $1+\Gamma_i$. As in the static setting, Γ_i is itself (the negative of) a generalized projection coefficient of the informed trader's order flow filter on his private signal against the total order flow.

The "denominator" of (33) is the analogue of the variance of that part of the price process that is driven by this forecast error. The solution for B_i in (33) is therefore the forecast error of the projection of the informed trader's information against the net information in total order flow.

Before developing the first-order condition for Γ_i the first-order condition for the market maker will be developed. That condition will be applied to simplify the informed trader's problem.

6.3. **The market-maker's objective.** The market-maker strives to minimize the forecast error variance of price conditional on order flow:

$$\max_{\{\Lambda\}} - \int_{a-i\infty}^{a+i\infty} \mathrm{tr} \left\{ \begin{pmatrix} \Phi - B_1 H\Lambda \\ \vdots \\ \Phi - B_N H \Lambda \\ -H\Lambda \end{pmatrix} \begin{pmatrix} \Phi^* - \Lambda^* H^* B_1^* & \ldots & \Phi^* - \Lambda^* H^* B_N^* & -\Lambda^* H^* \end{pmatrix} R \right\} ds$$

with first-order condition

$$\begin{pmatrix} -H^* B_1^* & \ldots & -H^* B_N^* & -H^* \end{pmatrix} R \begin{pmatrix} \Phi - B_1 H\Lambda \\ \vdots \\ \Phi - B_N H \Lambda \\ -H\Lambda \end{pmatrix} = \mathcal{A}^* \tag{34}$$

where the matrices have been transposed under the trace operator. Also, the two separate terms of the first-order condition have been consolidated into a single one by taking the conjugate-transpose of the second term. Defining the function J via the factorization

$$J^* J \equiv \begin{pmatrix} B_1^* & \dots & B_N^* & 1 \end{pmatrix} R \begin{pmatrix} B_1 \\ \vdots \\ B_N \\ 1 \end{pmatrix} = \begin{pmatrix} B^* & I \end{pmatrix} R \begin{pmatrix} B \\ I \end{pmatrix} = B^* R_e B + R_u. \tag{35}$$

it is possible to write the first-order condition as

$$H^* J^* J H \Lambda = H^* \begin{pmatrix} B^* & I \end{pmatrix} R \begin{pmatrix} \Phi \\ 0 \end{pmatrix} + \mathcal{A}^* = B^* R_e \Phi + \mathcal{A}^*.$$

where in the last step the block-diagonal structure of R has been used. Note also that the filter characterizing the total order flow process is JH.

Multiplying both sides by H^{*-1},

$$J^* J H \Lambda = B^{*\prime} R_e \Phi + \mathcal{A}^* \tag{36}$$

with solution

$$\Lambda = H^{-1} J^{-1} \left\{ J^{*-1} B^{*\prime} R \begin{pmatrix} \Phi \\ 0 \end{pmatrix} \right\}_+ \tag{37}$$

The interpretation of the solution in (37) is straightforward. The total order flow process process is implicitly defined by the filter J. The solution for Λ is then the s-transform analogue of the projection coefficient of the true value process on total order flow.

We now return to the first-order condition for Γ_i, which makes use of the market-maker's first-order condition (34).

6.3.1. *The informed trader's first-order condition for* Γ_i. The first-order condition for Γ_i is

$$\begin{pmatrix} B_1^* & \dots & B_N^* & 1 \end{pmatrix} R \begin{pmatrix} \Phi - B_1 H \Lambda \\ \vdots \\ \Phi - B_N H \Lambda \\ -H\Lambda \end{pmatrix} \tag{38}$$

$$+ \begin{pmatrix} -\Lambda^* B_1^* & \dots & -\Lambda^* B_N^* & -\Lambda^* \end{pmatrix} R \begin{pmatrix} B_1 \Gamma_i \\ \vdots \\ B_i (1 + \Gamma_i) \\ \vdots \\ B_N \Gamma_i \\ \Gamma_i \end{pmatrix} = \mathcal{A}^* \tag{39}$$

Substituting from the market-maker's first-order condition (34), the first term drops out, yielding

$$\begin{pmatrix} -\Lambda^* B_1^* & \dots & -\Lambda^* B_N^* & -\Lambda^* \end{pmatrix} R \begin{pmatrix} B_1\Gamma_i \\ \vdots \\ B_i(1+\Gamma_i) \\ \vdots \\ B_N\Gamma_i \\ \Gamma_i \end{pmatrix} = \mathcal{A}^*$$

Eliminating the Λ^* term yields

$$\begin{pmatrix} B^* & I \end{pmatrix} R \begin{pmatrix} B_1\Gamma_i \\ \vdots \\ B_i(1+\Gamma_i) \\ \vdots \\ B_N\Gamma_i \\ \Gamma_i \end{pmatrix} = \mathcal{A}^*.$$

Using the block-diagonal structure of R yields

$$J^* J \Gamma_i = -B^* R_e \begin{pmatrix} 0 \\ \vdots \\ B_i \\ \vdots \\ 0 \end{pmatrix} + \mathcal{A}^* \tag{40}$$

with solution

$$\Gamma_i = -J^{-1} \left\{ J^{*-1} B^{*\prime} R_e \begin{pmatrix} 0 \\ \vdots \\ B_i \\ \vdots \\ 0 \end{pmatrix} \right\}_+ \tag{41}$$

As was pointed out above, the solution (41) is the s-transform analogue of (the negative of) the projection coefficient of the informed trader's filter on his private information against the information in total order flow.

This fact can be used to interpret the informed trader's order flow strategy. Examining the informed trader's frequency domain objective in (32), the trader's order flow process is characterized by the vector of the filters

$$\begin{pmatrix} B_1\Gamma_1 & \dots & B_i(1+\Gamma_i) & \dots & B_N\Gamma_N \end{pmatrix}$$

acting on the vector of processes $\begin{pmatrix} e_1(t) & \dots & e_N(t) & u(t) \end{pmatrix}'$; the Γ_i terms express the projection on the information in price. Interpreting Γ_i as negative—as was the case in the static example—this projection is subtracted from direct trade process on the private information itself, that is from the filter B_i acting directly on the process $e_i(t)$. The interpretation is that the informed trader knows that any information the market makers can infer about his private information will be incorporated in price and thereby its profit potential neutralized. The informed trader thus trades only on the residual, unforecastable part of his private signal.

The formulas for B_i, Γ_i, and Λ can now be solved in examples. The main example posits that the value process follows an Ornstein-Uhlenbeck process, the continuous-time analogue of an autoregressive process.

7. Some propositions about the order flow and price processes in equilibrium: Inconspicuousness

As also demonstrated in [9], [8] and [28], the forecast error characterization of the trading strategies has broader implications. First, because the informed traders do not want to be detected by the market makers or by their rivals, they hide behind the noise traders. This requires that the order flow process have no dynamic structure that would allow market makers to infer the informed trades. Therefore, the total order flow will have the same stochastic structure as the noise trade process.

Second, the price process must not have a dynamic structure that is fundamentally different from the dynamic structure of the fundamental asset value process, as this would allow the informed traders to arbitrage against it purely based on filtering the dynamic structure.

Proposition 5. *The total order flow process filter*

$$JH$$

is a constant matrix. Therefore order flow has the same dynamic structure as the filter for the noise trade process $u(t)$*.*

Proof: Add up the Γ_i equations (40) across traders, yielding

$$J^*J\sum_i \Gamma_i = -B^*R_e\sum_i \begin{pmatrix} 0 \\ \vdots \\ B_i \\ \vdots \\ 0 \end{pmatrix} + \mathcal{A}^*$$

Recalling the definition $H \equiv \sum_i \Gamma_i + I$ and using the vector expression B,

$$J^*J(H-I) = -B^*R_eB + \mathcal{A}^*.$$

From the definition of J in equation (35),

$$J^*J(H-I) = -(J^*J - R_u) + \mathcal{A}^*$$

where R_u is the covariance function for the noise trade process. Also recall that it is assumed that the covariance functions for the fundamental processes are Dirac δ functions, so R_u is a constant matrix. Because no further filter is applied to the noise trade process, noise trade is white noise.

Algebraic manipulation then yields

$$J^*JH = R_u + \mathcal{A}^*.$$

Order flow is then

$$JH = \left\{ {J^*}^{-1} R_u \right\}_+$$

Because R_u is a constant matrix, the projection operator eliminates all poles in the negative half plane. But J was constructed via factorization to have poles only in the positive half plane, and therefore the projection must be a constant. □

Proposition 6. *The price process filter*

$$JH\Lambda$$

has the same pole structure as the value process filter Φ*, and therefore the price process has the same dynamic structure as the value process.*

Proof: The filter for the price process is the order-flow filter JH multiplied by the pricing filter Λ.

Multiplying the first-order condition for Λ, (36) by ${J^*}^{-1}$ and applying the annihilator yields the equation for the price process,

$$JH\Lambda = \left\{ {J^*}^{-1} B^* R_e \Phi \right\}_+ .$$

By Lemma 9 (see the Appendix), the right hand side is the product of a constant matrix and Φ. Thus,

$$JH\Lambda = CR_e\Phi. \tag{42}$$

where C is a constant matrix $J(r+\rho)^{-1}B(r+\rho)$. □

7.1. **Acceleration.** As was shown in [28] and [9], the informed traders trade intensely on their information, in the sense that the filter on the fundamentals of their private signals has a pole structure such that their order flow on private signals is less serially correlated than the asset value itself. The proof of this was set out in [28] for the discrete time multi-asset case. The proof there has two main parts. The first part is to establish that the poles exceed the poles of Φ. This is done by showing that the equilibrium mapping of a conjectured pole structure for B_i results in a set of larger poles. The second part follows by showing that the number of poles increases by one for each iteration of the mapping, and that therefore there must be infinitely many in equilibrium.[16]

8. Discussion and related literature

The results here are a little more general than some of the literature in the following sense. In Danilova [14] for example, which has dynamic evolution of the fundamental value and a single informed trader, the hiding idea is termed *inconspicuousness*. But in that model, the asset fundamental value process and the noise trade are both Brownian processes with jumps, so the dynamic structure or order flow is no different than the price process. Here, by contrast, the noise trade is serially uncorrelated while the fundamental value process is serially correlated. Therefore in order to hide, the informed traders must adjust not just the magnitudes of their trades, but their dynamic pattern.

[16] To clarify terminology, a pole can be intuitively viewed as the inverse of an autoregressive coefficient in discrete time, and is therefore (for stationary processes in the discrete-time setting of [28]) greater than the square root of the discount factor in absolute value. In discrete time poles correspond to points in the right half plane. As the number of poles is infinite, this requires that the poles converge to infinity. It can then be argued that arbitrary rational functions can be approximated by the sums of such pole terms, and also characterized by the pattern of weights on those pole terms.

8.1. **Relationship with the order-splitting literature.** The reason for the order flow result is related to the result of Back and Baruch [4]: the informed traders pool with the noise traders, that is, they hide their trades. The Back and Baruch model establishes that breaking up large block orders into a sequence of small ones is optimal, but the result here emphasizes that it is not the breaking up of the orders that is crucial, but the fact that the orders are stochastically indistinguishable from the noise trades that matters.

In this sense the model also suggests that there is not an important difference between dealership markets and other market structures such as a limit order market, buttressing Back and Baruch's central finding.

Back and Baruch [4] set out a model in which informed traders can post orders of any size; in equilibrium they order one share at a time, with large (block) orders being expressed as a high rate of single-share orders. The result is that informed traders pool with uninformed traders. The results here are equivalent: informed traders want to appear like noise traders, otherwise their information can be extracted by market makers. Back and Baruch demonstrate the equivalence of their market structure with one in which there is an open (public) order book with limit orders, in which informed traders put in limit orders.

In particular, Back and Baruch note that in a floor-trading model—that is, one with competitive market makers, as in the Kyle [24] model, the informed traders might submit a large order, but it must be structured (via a mixed strategy) so that market makers cannot clearly identify it as an informed order as would be the case in a separating equilibrium:

> When orders are worked, liquidity providers on a floor exchange can of course condition on the size of an order, but they cannot condition on the size of the demand underlying the order—they cannot know whether there will be more orders from the same trader in the same direction immediately forthcoming. Thus, in a pooling equilibrium on a floor exchange, ask prices are upper-tail expectations—expectations conditional on the size of the demand being the size of the order or larger—precisely as in a limit-order market. This is the reason a pooling (worked-order) equilibrium on a floor exchange is equivalent to a block-order equilibrium in a limit-order market. ([4], p. 2)

8.2. **Relationship to the speed of revelation literature.** The high intensity of trading relative to information arrival result matches the similar result from the discrete-time models in Bernhardt, Seiler and Taub [9] and [28], also expressed by the infinitely long pattern of poles. There is a related result in the paper of Chau and Vayanos [13]. The Chau and Vayanos model uses a different information structure: market makers can observe firm value contemporaneously, but cannot forecast its evolution, while the informed trader can forecast. In their model, which has a single informed trader in an infinite horizon setting with stationary asset value evolution and continuous information arrival, trading intensity is accelerated relative to the arrival rate of information, but because of the model structure the acceleration results in full and immediate revelation of the informed trader's information.

9. The dynamic model with a large shareholder

The large shareholder chooses the filter $\theta_1(\cdot)$ on his private signal in order to alter fundamental value:

$$v_t = \int_0^\infty \Big(\phi(\tau) \sum e_{i,t-\tau} - \theta_1(s)\phi(\tau)e_1(t-\tau)\Big)\, d\tau$$

There are no intrinsic restrictions on the filter, other than that it must by analytic, i.e., it can be backward looking but not forward looking. Thus, the filter can be designed so that the stochastic structure in the underlying shocks is altered to be more or less persistent, and to have additional structure such as zeroes that make the price process noninvertible in some appropriate sense.

The penalty on the large shareholder's action is

$$E\left[\left(C^{1/2}\int_0^\infty \theta_1(s)\phi(s)e_{1,t-s}ds\right)^2\right],$$

that is, the amplification is treated as a penalty process.

9.1. **The large shareholder's problem.**

There are N informed traders. Each of the informed traders privately observes one signal process, represented here by Φ_i. Trader 1 is the large shareholder who has not only has private information about one of the processes that affect fundamental value, he can affect that process. He chooses a filter, Θ_1, to potentially alter not only the magnitude of the fundamental process Φ_1, but also potentially its dynamic (i.e. autoregressive) structure.

The informed shareholder's objective is a modification of the objective in the stock market model up to now: instead of

$$\max_{X_i} \int_{a-i\infty}^{a+i\infty} (\Phi(s) - \Lambda(s)O(s))X_i^*(r-s^*)Rds$$

where O is total order flow (the sum of informed trades $\sum_i x_{it}$ and noise trade orders u_t), and where $\Phi(s)$ is the (vector) value process Laplace transform, the valuation process reflects the action of the large shareholder:

$$\max_{X_1,\Theta_1} \int_{a-i\infty}^{a+i\infty} \Big(\Phi(s) - \Theta_1(s)\Phi_1(s)) - \Lambda(s)O(s))X_1^*(r-s^*)R - \frac{C}{2}\Theta_1(s)\Phi(s)\Phi^*(r-s^*)\Theta_1^*(r-s^*)\Big)ds \tag{43}$$

For compactness I will denote the value process after manipulation by the large shareholder by $\tilde{\Phi}$. The large shareholder acts on his private signal which has filter Φ_1, so the modified value process filter might be

$$\tilde{\Phi} = \begin{pmatrix}\Phi_1(s) - \Theta_1(s)\Phi_1(s) & \Phi_2(s) & \dots & \Phi_N(s)\end{pmatrix}'$$

that is, only the first component of firm value is directly affected by the actions, which is done via his choice of Θ_1, the Laplace transform of the large shareholder's filter on his fundamental signal $\Phi_1(s)$; C is a constant measuring the cost of the action filter Θ_1.

Breaking out the pieces of the objective yields

$$\max_{\{B_1,\Gamma_1,\Theta_1\}} -\int_{a-i\infty}^{a+i\infty} \operatorname{tr}\left\{ \begin{pmatrix} (\Phi_1-\Theta_1\Phi_1)-B_1H\Lambda \\ \vdots \\ \Phi_N - B_N H\Lambda \\ -H\Lambda \end{pmatrix} \begin{pmatrix} (1+\Gamma_1^*)B_1^* & \cdots & \Gamma_1^* B_i^* & \cdots & \Gamma_1^* B_N^* & \Gamma_1^* \end{pmatrix} R - \frac{C}{2}\Theta_1\Phi_1\Phi_1^*\Theta_1^* \right\} ds \tag{44}$$

for the informed large shareholder.

9.1.1. *The large shareholder's first order conditions.* The first-order condition for B_i will now have some extra terms due to the penalty function, and there is an entirely new first-order condition for Θ_1.

Assuming no cross-correlation of information, the first-order condition for B_1 is

$$\Big[((\Phi_1-\Theta_1\Phi_1)-B_1H\Lambda)(1+\Gamma_1^*) - B_1(1+\Gamma_1)\Lambda^*H^* \Big] R_e = \mathcal{A}^* \tag{45}$$

This first-order condition is a straightforward modification of the corresponding condition in the pure trading model: defining G_1 by

$$G_1G_1^* \equiv [\Lambda H(1+\Gamma_1^*) + (1+\Gamma_1)H^*\Lambda^*]R \tag{46}$$

the solution is

$$B_1 = G_1^{-1}\left\{ G_1^{*-1}(\Phi_1-\Theta_1\Phi_1)(1+\Gamma_1^*)\right\}_+ \tag{47}$$

Thus, the first-order condition for filtering private signals is identical in structure to that for the pure exchange model of [9] and [28]. That is, the large shareholder's production-control decisions appear here only as a modification of the effective signal process. In this sense the production control decision is separable from the trading strategy decision.

If Θ_1 is a scalar constant, then the overall equilibrium will be like that of the pure exchange model, with the large shareholder amplifying the unforecastable part of the fundamental. On the other hand, if Θ_1 has a nontrivial filter structure, then the production-control decisions will affect the large shareholder's trading strategy filter as well.

9.1.2. *The first order condition for* Γ_1*.* The first order condition for Γ_1 is similar: it is the same as for the informed traders who are not large shareholders, but with the value process replaced by the large-shareholder-modified process.

9.1.3. *The first order condition for* Θ_1*.* I now turn to the first order condition for Θ_1, the large shareholder's amplification factor. It already emerged in the static model that the large shareholder amplifies the variance of his component of firm value. I now show that he also alters its time series structure.

The first order condition is (with off-diagonal terms dropping out due to the trace operation)

$$-\Phi_1^*\left(B_1(1+\Gamma_1)\right)R_e - (\Theta_1\Phi_1\Phi_1^* + \Phi_1\Phi_1^*\Theta_1^*)\frac{1}{2}C = \mathcal{A}^* \tag{48}$$

Consolidating via the conjugate transpose yields

$$-\Phi_1^* \left(B_1(1+\Gamma_1)\right) R_e - \Theta_1\Phi_1(s)\Phi_1^* C = \mathcal{A}^* \tag{49}$$

Dividing out Φ^* then yields

$$-\left(B_1(1+\Gamma_1)\right) R_e - \Theta_1\Phi_1(s) C = \mathcal{A}^*$$

The solution of the first-order condition for Θ_1 (49) is then straightforward:

$$\Theta_1 = -C^{-1}B_1(1+\Gamma_1)R\Phi_1^{-1}$$

When we substitute this into the objective, the Φ^{-1} term cancels the Φ term in the objective, leaving

$$\tilde{\Phi} = \Phi + B_1(1+\Gamma_1)C^{-1}.$$

As with the static model, the term that is added on is simply the unforecastable part of the large shareholder's trade!

It is also worth noting that even though the model was not set up to require that the large shareholder's ability to affect the fundamental is proportional to his holdings or trades, this nevertheless emerges endogenously, that is, Θ is proportional to B_1, the large shareholder's trade on his private information.

9.1.4. *Demonstrating the non-constancy of* Θ*.* To demonstrate the non-constancy of Θ, I make the following assumptions:

Assumption 7. (i) *The fundamental process Φ has only one pole*
(ii) *The correlation between the fundamental and the noise trade process is zero, that is, R is diagonal*
(iii) *There is only one stock*
(iv) *There is only one privately informed trader, namely the large shareholder*

These assumptions are all standard in relation to the simplest Kyle models. If Θ_1 were in fact constant, then the large shareholder would simply amplify the fundamental process as in the static model, that is, the fundamental would become $(1-\Theta_1)\Phi$, with Θ_1 negative.

Proposition 8. *Let Assumptions 7 be met. Then Θ_1 is not a constant and therefore $(1-\Theta_1)\Phi_1$ is not proportional to Φ_1, that is, the large shareholder alters the autoregressive structure of the firm's fundamentals.*

Proof: The agenda is to demonstrate that $(1+\Gamma_1)B_1\Phi_1^{-1}$ is not of order Φ_1. Recalling the solution for B_1, we have

$$G^{-1}\left\{G^{*-1}(1-\Theta_1)\Phi(1+\Gamma_1^*)\right\}_+(1+\Gamma_1)$$

Now apply the annihilator lemma, Lemma 9, yielding

$$\sim G^{-1}\Phi(1+\Gamma_1)$$

To establish that this expression not of order Φ_1, we can equivalently demonstrate that $\sim G^{-1}(1+\Gamma_1)$ is not a constant. Recalling the definition of G from equation (46),

$$|G_1|^2 = G_1G_1^* \equiv [\Lambda H(1+\Gamma_1^*) + (1+\Gamma_1)H^*\Lambda^*] \tag{50}$$

Thus,

$$\begin{aligned}\left|(1+\Gamma_1)^{-1}G_1\right|^2 &= (1+\Gamma_1)^{-1}\left(\Lambda H(1+\Gamma_1^*)+(1+\Gamma_1)H^*\Lambda^*\right)(1+\Gamma_1^*)^{-1}\\ &= (1+\Gamma_1)^{-1}\Lambda H + H^*\Lambda^*(1+\Gamma_1^*)^{-1}\end{aligned}$$

Now recall that for $N=1$, that is, the assumption that there is only one informed trader, namely the large shareholder, $H = 1+\Gamma_1$; thus, assuming we have commutativity, this expression reduces to

$$\Lambda + \Lambda^* \equiv |L|^2$$

We know that Λ is of order Φ from Proposition 6; this sum therefore defines a process $L(s)$ that has nontrivial zeroes as well as the same poles as Φ. Thus, $L(s)$ is the product $N(s)\Phi(s)$, and $N(s)$ is a non-constant function. Therefore $G^{-1}(1+\Gamma_1) = L^{-1}$ is a non-constant function, and so $G^{-1}(1+\Gamma_1)\Phi$ is proportional to $N(s)^{-1}$, which is not proportional to Φ. □

The result extends if there are multiple informed traders with only one large shareholder.

Thus, we can conclude that the large shareholder *dynamically* obfuscates: he not only amplifies the fundamental value process, he alters its time series structure by the market makers' forecast error.

Finally, it should be noted that the analysis above suppresses the interaction between Θ_1 and B_1 in the first-order conditions. However, it is possible to state a joint first order condition in vector form: combining (45) and (49) yields the vector condition, we have

$$\begin{aligned}\begin{pmatrix} H\Lambda(1+\Gamma_1^*)+(1+\Gamma_1)\Lambda^*H^* & \Phi_1(1+\Gamma_1^*) \\ (1+\Gamma_1)\Phi_1^* & C\Phi_1\Phi_1^*\end{pmatrix}\begin{pmatrix} B_1 \\ \Theta_1\end{pmatrix} \\ = \begin{pmatrix}\Phi_1(1+\Gamma_1^*) \\ 0\end{pmatrix} + \mathcal{A}^*\end{aligned} \tag{51}$$

Note the endogenous terms on the right. The key observation though is that the coefficient matrix on the left is Hermitian and so can be factored, making possible a conventional solution of the model. This extension is carried out in Appendix E, and an example is computed.

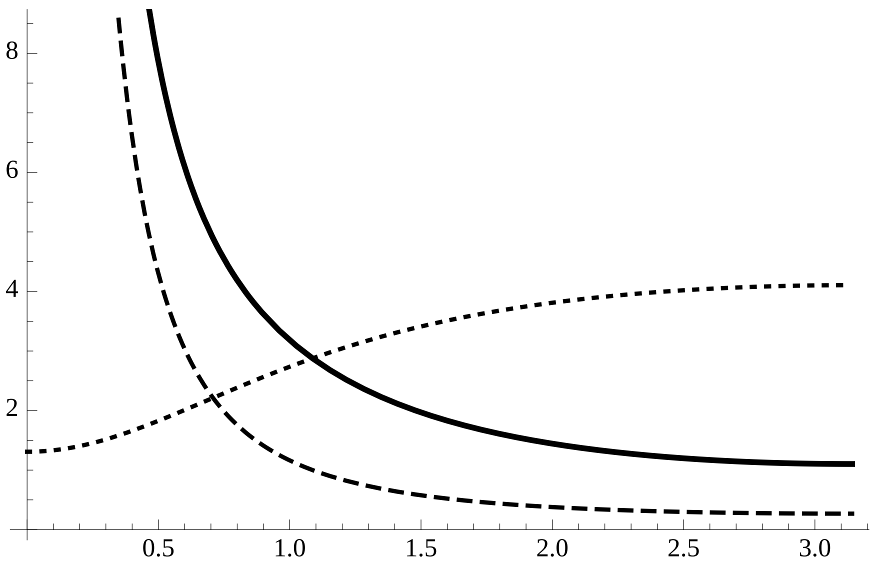
Plotting the spectral density of the Φ and $(1-\Theta)\Phi$ filters in the example shows that it amplifies high frequencies more than low frequencies—that is, it actually reduces the persistence of the fundamental. The intuition for the emphasis on high frequencies is that the large shareholder wants to masquerade as a noise trader (that is, to become even more inconspicuous), and the noise trade has zero persistence. Thus, the amplification factor moves the fundamental in the direction of white noise. Note also that the amplification factor increases the level of the spectral density at every frequency—that is, the overall volatility of the fundamental is increased, as the analytical model demonstrates.

10. Conclusion

The following phenomena were demonstrated:

(i) The large shareholder amplifies the unforecastable part of the fundamental in order to increase his trading profits.

Figure 1. Plot of the spectral densities



This figure plots the spectral densities of the Φ process filter $(1 - .93z)^{-1}$ (dashed line), the insider's amplification filter $(1 - \Theta_1)$ (dotted line) and the net process filter (solid line). The amplification filter amplifies high frequencies more than low frequencies, thus reducing the persistence of the fundamental.

(ii) In a dynamic setting, informed traders jointly behave so as to be inconspicuous, so that total order flow has the autoregressive structure of the noise trade, and price has the autoregressive structure of the fundamental value process.
(iii) In a dynamic setting, the large shareholder alters the autoregressive structure of the firm's fundamental value process, that is, he dynamically obfuscates, and because this alteration is based on the market makers' forecast error process, it does not increase the amount of information available to the market.

Given that the large shareholder's trading profits can be enhanced by his alteration and amplification of the private signal, there is an incentive to acquire private information, along the lines set out in [7]. One usually unspoken element of private information models is the reason for the privacy or unobservability of the information. It is evident here that there are strong incentives to acquire private information and also to keep it private.

A central feature of business cycles is that they are persistent relative to the shocks that induce them. Successfully explaining this persistence requires explaining how firms fail to adjust quickly to shocks. The model here suggests that agents in possession of private information about fundamental shocks will not only obfuscate that information by amplifying the unforecastable part, they will add to the obfuscation by deliberately altering its autoregressive structure. Thus, if there is any uncertainty about aggregate nominal processes along the lines of [25], such obfuscation will actually exacerbate the associated signal extraction problem, and with it, deliver the aggregate fluctuations we observe.

Appendix A. Fourier transforms of continuous-time processes

The Ornstein-Uhlenbeck process is the continuous-time analogue of the discrete autoregressive process:

$$dx = axdt + dz$$

The integral representation is

$$x(t) = \int_{-\infty}^{t} e^{a(t-s)} dz(t)$$

The Fourier transform of $e^{a(t-s)}$ is

$$\frac{1}{i\omega - a}$$

and the Fourier transform of dz (which corresponds to white noise) is the δ function. (A reference is Igloi and Terdik [22], p. 4.) The spectral density is

$$\frac{1}{a^2 + \omega^2}$$

The Fourier transform (and corresponding Laplace transform) resemble the pole forms $1/(z - a)$ in discrete time models. The building block in the s-domain is therefore also rational functions, except that causality is associated with poles in the left half plane instead of the unit circle.

Observe that $a = 1$ yields the Fourier transform of a standard Brownian motion; thus, with discounting it isn't a problem to translate standard discrete-time stationary models to this setting, and vice versa.

Appendix B. Practical details of spectral factorization and annihilator operations in continuous time

In this appendix I examine how factorization and the annihilation operator work in practical examples. I begin by briefly recapitulating an example from Kailath, Sayed and Hassibi, p. 263-264. Kailath, Sayed and Hassibi posit a model which has the following Wiener-Hopf equation:

$$K(s)S_y(s) = S_{sy}(s)e^{s\lambda} - G(s) \tag{KSH 7.A.4}$$

Here $K(s)$ is the Laplace transform (s-transform) of the unknown filter that is to be found; $G(s)$ is the Laplace transform of a function $g(t)$ that is a purely anticausal function, that is, a function that is analytic on the left half plane only and zero in the right half plane, but which is otherwise arbitrary, corresponding to the principal part function $\sum_{-\infty}^{-1}$ in the discrete time setting: $g(t) = 0$, $t > 0$; $S_{sy}(s)$ and $S_y(s)$ are the Laplace transforms of variance and covariance functions

$$S_y(s) = \mathcal{L}\{R_y\} \qquad S_{ys}(s) = \mathcal{L}\{R_{ys}\}$$

with

$$R_y(\tau) \equiv E[\mathbf{y}(t)\mathbf{y}(t-\tau)] \qquad R_{ys}(\tau) \equiv E[\mathbf{s}(t)\mathbf{y}(t-\tau)]$$

Note that Kailath, Sayed and Hassibi have some contrasting notation: process $\mathbf{s(t)}$ is in boldface, and the argument of the Laplace-transformed function s, which is completely different. Thus, S_y is the Laplace transform of the observed process, and $\mathbf{s}(t)$ is the signal process that the observer wants to extract; R_{ys} is then the covariance function between the observed and signal processes.

The exponential term appears in the Wiener-Hopf equation because the original equation is shifted:

$$R_{sy}(t+\lambda) = \int_0^\infty k(\tau) R_y(t-\tau) d\tau, \qquad t > 0$$

which captures the idea of time-lagged observations.

To solve the problem ([23] 7.A.4), first factor S_y. Abstractly, this factorization is

$$S_y(s) = L(s) R L^*(-s^*) \tag{KSH 7.A.2}$$

where R is a positive constant, and $L(s)$ is causal, that is, both L and L^{-1} are analytic on the right half plane.

Now write the solution:

$$K(s) = L(s)^{-1} \left\{ L^*(-s^*)^{-1} R^{-1} S_{xy}(s) e^{s\lambda} \right\}_+ \tag{KSH 7.A.7}$$

The remaining agenda is to carry out a factorization for a practical problem and to demonstrate how the annihilation operation works in that practical setting.

Kailath, Sayed and Hassibi posit a signal process with Fourier transform spectral density

$$S_s(f) = \mathcal{F}\left\{ e^{-\alpha|t|} \right\} = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

Note that there is a distinction between the Fourier and Laplace representations. Defining $s \equiv 2\pi i f$, the equivalent bilateral Laplace transform is

$$S_s(s) = \mathcal{L}\left\{ e^{-\alpha|t|} \right\} = \frac{2\alpha}{\alpha^2 - s^2}$$

The noise process $\mathbf{v}(t)$ is white noise (not the same as a Brownian process!) which has a flat spectrum:

$$S_v(s) = 1$$

and the sum of the signal and noise, $\mathbf{y}(t) = \mathbf{s}(t) + \mathbf{v}(t)$, is (because the Laplace transform of a sum is the sum of the Laplace transforms)

$$S_y(s) = S_s(s) + S_v(s) = \frac{2\alpha}{\alpha^2 - s^2} + 1 = \frac{2\alpha}{\alpha^2 - s^2} + 1 = \frac{s^2 - \alpha^2 - 2\alpha}{s^2 - \alpha^2} = L(s) R L^*(-s^*).$$

The denominator of this expression is the product $(s-\alpha)(s+\alpha)$. Restate the entire expression as a product:

$$\frac{s + \sqrt{\alpha^2 + 2\alpha}}{s + \alpha} \frac{s - \sqrt{\alpha^2 - 2\alpha}}{s - \alpha}$$

so that

$$L(s) = \frac{s + \sqrt{\alpha^2 + 2\alpha}}{s + \alpha}.$$

(Of course this is just one of the potential factorizations.) Note that L is analytic in the right half plane because its pole, $-\alpha$, is in the left half plane, and the inverse is analytic in the right half plane because the zero, $-\sqrt{\alpha^2 + 2\alpha}$, is in the left half plane.

The final step is to calculate the annihilate. To do this, a partial fractions calculation must be done. The argument of the annihilator is

$$\frac{s-\alpha}{s-\sqrt{\alpha^2+2\alpha}}\frac{2\alpha}{\alpha^2-s^2}$$

Writing out the factors in the denominator, there is a cancellation:

$$=\frac{s-\alpha}{s-\sqrt{\alpha^2+2\alpha}}\frac{2\alpha}{(\alpha-s)(\alpha+s)}=-\frac{1}{s-\sqrt{\alpha^2+2\alpha}}\frac{2\alpha}{\alpha+s}.$$

Now rewrite this with partial fractions:

$$=\frac{-\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{s-\sqrt{\alpha^2+2\alpha}}+\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{\alpha+s}.$$

The annihilator kills elements that have poles in the right half plane; the first term will therefore be killed:

$$\left\{\frac{-\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{s-\sqrt{\alpha^2+2\alpha}}+\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{\alpha+s}\right\}_+=\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{\alpha+s}.$$

Therefore the solution of the Wiener-Hopf equation is

$$K(s)=\frac{s+\alpha}{s+\sqrt{\alpha^2+2\alpha}}\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{\alpha+s}=\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{s+\sqrt{\alpha^2+2\alpha}}.$$

This is the Laplace transform for a filter. The actual filter can be obtained by performing the inverse transform operation.

B.1. **A small lemma about the annihilator.** The annihilator operator is a linear operator and therefore can be expressed as an integral ([23], p. 263):

$$\{F(s)\}_+=\int_0^\infty\left[\frac{1}{2\pi i}\int F(p)e^{pt}dp\right]e^{-st}dt \tag{52}$$

The interpretation is straightforward: perform the inverse Laplace transform with the inner integral (which in conventional situations is integrated along the imaginary axis). Then perform the one-sided Laplace transform on the result, which picks up only the part of the function defined for positive t, that is, in the right half plane. The following small lemma holds, which is a variation of Whittle's theorem.

Lemma 9. *Let F be analytic in the right half plane, and let $a>0$. Then*

$$\left\{F^*(r-s^*)\frac{1}{s+a}\right\}_+=F(r+a)\frac{1}{s+a}$$

Proof: I will first demonstrate this for a simple version of F, namely $F(s)=\frac{1}{s+b}$, $b>0$—namely when F is also the filter for an Ornstein-Uhlenbeck process. In that case, the inner integral of (52) is

$$\frac{1}{2\pi i}\int\frac{1}{-p+r+b}\frac{1}{p+a}e^{pt}dp$$

Now do partial fractions:

$$=\frac{1}{2\pi i}\int\left(\frac{\frac{1}{r+b+a}}{-p+r+b}+\frac{\frac{1}{r+b+a}}{p+a}\right)e^{pt}dp$$

The integration is along the imaginary axis. This is equivalent (via a Möbius transform) to integrating around the unit circle. Consequently Cauchy's theorem can be invoked: a holomorphic function with a pole in the right half plane integrates to zero. The pole of the first term in the expression is $p+r$, and therefore the integral of the first term is zero. The remaining expression is

$$\frac{1}{r+b+a}e^{-at}$$

Now take the outer integral.

$$\int_0^\infty \left[\frac{1}{r+b+a}e^{-at}\right] e^{-st}dt = \frac{1}{r+b+a}\frac{1}{s+a}.$$

This completes the proof for this simple case.

If f is analytic, then it can be represented in power series form:

$$f(\tau) = \sum_{k=0}^{\infty} f_k e^{-b_k \tau}.$$

The s-transform of this function is

$$F(s) = \sum_{k=0}^{\infty} f_k \frac{1}{s+b_k}.$$

Now proceed as in the proof above for each k. □

This result is stated and proved in greater generality for matrix systems in [31] using state space methods. When general compound expressions of the sort $\{FG^*\}_+$, where both F and G are analytic, that is, their poles are in the left half plane, are viewed from a state space perspective, it is clear that the product will be a function with poles in the left half place inherited from the poles of F and poles in the right half plane inherited from G. The annihilator removes the latter poles, while the poles of F survive.

Appendix C. Factoring the coefficient matrix

The coefficient matrix in the matrix representation of the stacked first order conditions, equation (51), can be written as

$$\begin{pmatrix} (1+\Gamma_1^*) & 0 \\ 0 & \Phi^* \end{pmatrix} \begin{pmatrix} (1+\Gamma_1)^{-1}H\Lambda + \Lambda^* H^*(1+\Gamma_1^*)^{-1} & 1 \\ 1 & C \end{pmatrix} \begin{pmatrix} (1+\Gamma_1) & 0 \\ 0 & \Phi \end{pmatrix} \tag{53}$$

The internal matrix is then easier to factor because only the upper left element is nonscalar.

Recalling the definition of H from equation (6), because $N = 1$, we have that $H = 1 + \Gamma_1$, and therefore from equation (53) we have:

$$\begin{pmatrix} (1+\Gamma_1^*) & 0 \\ 0 & \Phi^* \end{pmatrix} \begin{pmatrix} \Lambda + \Lambda^* & 1 \\ 1 & C \end{pmatrix} \begin{pmatrix} (1+\Gamma_1) & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} B_1 \\ \Theta_1 \end{pmatrix} = \begin{pmatrix} \Phi_1(1+\Gamma_1^*) \\ 0 \end{pmatrix} + \mathcal{A}^* \tag{54}$$

and then use the factorization of the inner part:

$$\begin{pmatrix} (1+\Gamma_1^*) & 0 \\ 0 & \Phi^* \end{pmatrix} \bar{F}^* \bar{F} \begin{pmatrix} (1+\Gamma_1) & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} B_1 \\ \Theta_1 \end{pmatrix} = \begin{pmatrix} \Phi_1(1+\Gamma_1^*) \\ 0 \end{pmatrix} + \mathcal{A}^* \tag{55}$$

We can invert the outer parts and then the inner parts:

$$\bar{F}\begin{pmatrix}(1+\Gamma_1) & 0\\ 0 & \Phi\end{pmatrix}\begin{pmatrix}B_1\\ \Theta_1\end{pmatrix} = \bar{F}^{*-1}\begin{pmatrix}\Phi_1\\ 0\end{pmatrix} + \mathcal{A}^* \tag{56}$$

The cancellation of the $(1+\Gamma_1^*)$ on the right assumes that this is a scalar and invertible quantity. The solution is

$$\begin{pmatrix}B_1\\ \Theta_1\end{pmatrix} = \begin{pmatrix}(1+\Gamma_1) & 0\\ 0 & \Phi\end{pmatrix}^{-1}\bar{F}^{-1}\left\{\bar{F}^{*-1}\begin{pmatrix}\Phi_1\\ 0\end{pmatrix}\right\}_+$$

Now recall from Proposition 5 that $1+\Gamma_1 = H \sim J^{-1}$. Thus,

$$\begin{pmatrix}B_1\\ \Theta_1\end{pmatrix} = \begin{pmatrix}c_2 J & 0\\ 0 & \Phi^{-1}\end{pmatrix}\bar{F}^{-1}\left\{\bar{F}^{*-1}\begin{pmatrix}\Phi_1\\ 0\end{pmatrix}\right\}_+$$

where c_2 is a constant that can be derived from Proposition 5. Taking this a step further we have

$$\begin{pmatrix}B_1\\ \Theta_1\end{pmatrix} = \begin{pmatrix}c_2 J & 0\\ 0 & \Phi^{-1}\end{pmatrix}\bar{F}^{-1}\left\{\begin{pmatrix}\bar{F}_{11}^{*-1}\Phi\\ \bar{F}_{21}^{*-1}\Phi\end{pmatrix}\right\}_+$$

Recalling that $\Phi(s)$ is Ornstein-Uhlenbeck (the continuous time analogue of autoregressive) $\frac{1}{s+a}$, and invoking the annihilator theorem (Lemma 9 in Appendix B)

$$\begin{pmatrix}B_1\\ \Theta_1\end{pmatrix} = \begin{pmatrix}c_2 J & 0\\ 0 & \Phi^{-1}\end{pmatrix}\bar{F}^{-1}\begin{pmatrix}\bar{F}_{11}(r+a)^{-1}\Phi\\ \bar{F}_{12}(r+a)^{-1}\Phi\end{pmatrix} \tag{57}$$

When the algebra is carried further, the Φ terms will cancel from the solution for Θ_1. Only the factor $\bar{F}$ then influences Θ_1 directly. The exact structure of $\bar{F}$ can be used to show that Θ_1 is not a constant.

Now we can numerically calculate F and the inverses, products and annihilates.

Appendix D. Extending the Cholesky decomposition to matrices of rational functions

In this appendix I detail how to extend the Cholesky decomposition from ordinary matrices to rational functions. The agenda is to develop the candidate factor, but not require that the candidate factor be analytic and invertible.

One begins with a Hermitian $n \times n$ matrix H, with elements h_{ij}, that is to be factored. The immediate question is whether to right- or left-factor H:

$$H = LL^* \quad \text{left factor} \qquad \text{or} \qquad H = R^*R \quad \text{right factor}$$

I will follow the left factor strategy. This is sufficient, even if we want a right factorization: first observe that if H is Hermitian, then so is the transpose H'. Thus,

$$H' = (LL^*)' = (L^*)'L'$$

which is a right factorization of H', and

$$H' = (R^*R)' = R'(R^*)'.$$

which is a left factorization. Thus, to obtain a right factorization of H, we just need to find a left factorization of H' and take the transpose of the result R'. Thus, it is sufficient to consider the left factorization; the right factor is simply the transpose of the left factor of H'.

Next, begin the factorization. Note: this is highly parallel with the development on the Wikipedia page on the Cholesky decomposition for ordinary matrices. The algorithm is recursive. In the first step, we have

$$H^1 \equiv H.$$

At step i of the algorithm there is an intermediate matrix,

$$H^i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & a_{ii} & b_i^* \\ 0 & b_i & B_i \end{pmatrix}$$

where I_i is the i-dimensional identity, a_{ii} is the ith diagonal entry from H_i, b_i is the $(n-i) \times 1$ column vector and the block matrix B_i is the lower right $(n-i) \times (n-i)$ submatrix from H.

When H is a matrix of numbers, then we take the square root of a_{ii}, and construct the matrix

$$L_i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \sqrt{a_{ii}} & 0 \\ 0 & \frac{1}{\sqrt{a_{ii}}} b_i & I_{n-i} \end{pmatrix}$$

The H_2 operation equivalent to the square root is spectral factorization. Thus, we want to find f_i such that

$$a_{ii} = f_i^* f_i$$

This is relatively straightforward because by construction a_{ii} is a scalar function. Thus in the spectral factorization case,

$$L_i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & f_i & 0 \\ 0 & f_i^{*-1} b_i & I_{n-i} \end{pmatrix}$$

Now define

$$H_{i+1} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B_i - a_i^{-1} b_i b_i^* \end{pmatrix}$$

and

$$H_{n+1} = I_n$$

Then

$$L = L_1 L_2 \cdots L_n.$$

There is one detail: in the H_2 case, how do we know that

$$L_i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & f_i & 0 \\ 0 & f_i^{*-1} b_i & I_{n-i} \end{pmatrix} \qquad \text{not} \qquad \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & f_i^* & 0 \\ 0 & f_i^{-1} b_i & I_{n-i} \end{pmatrix} \qquad ?$$

In the $n = 2$ case, we can do the multiplication. First,

$$H_1 = H = \begin{pmatrix} h_{11} & h_{21}^* \\ h_{21} & h_{22} \end{pmatrix},$$

$$L_1 = \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & 1 \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} 1 & 0 \\ 0 & h_{22} - h_{11}^{-1} h_{21} h_{21}^* \end{pmatrix}$$

with

$$f_1 f_1^* = h_{11}$$

Defining g by the factorization

$$g g^* \equiv h_{22} - h_{11}^{-1} h_{21} h_{21}^*.$$

Then

$$L_2 = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$$

The factor is then

$$L = L_1 L_2 = \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & g \end{pmatrix}$$

so

$$\begin{aligned} H = LL^* &= \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & g \end{pmatrix} \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & g \end{pmatrix}^* \\ &= \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & g \end{pmatrix} \begin{pmatrix} f_1^* & h_{21}^* f_1^{-1} \\ 0 & g^* \end{pmatrix} \\ &= \begin{pmatrix} f_1 f_1^* & h_{21}^* \\ h_{21} & h_{21}^* f_1^{-1} f_1^{*-1} h_{21} + g g^* \end{pmatrix} \\ &= \begin{pmatrix} h_{11} & h_{21}^* \\ h_{21} & h_{22} \end{pmatrix} \end{aligned}$$

Once this initial factor L is constructed, the Ball-Taub algorithm [2] can be applied to convert the factor to analytic and invertible form. That algorithm is presented under that assumption that a preliminary *right* factor R has been found and the appropriate adjustment must be made.

Appendix E. Factoring an example

Let us consider a simple example in which the fundamental firm value process is an AR(1) process and expressed by

$$\Phi(z) = \frac{1}{1 - az}.$$

We know from previous reasoning that this will lead to the pricing filter to have the same structure as the fundamental process, that is,

$$\Lambda = \frac{\lambda}{1 - az}.$$

For that reason, the matrix to be factored is

$$H \equiv \bar{F}^* \bar{F} = \begin{pmatrix} \Lambda + \Lambda^* & 1 \\ 1 & C \end{pmatrix}$$

In the following example, $a = .93$ (so $a^{-1} = 1.075$) and $C = 1.16$. We want to find a right factor, but the algorithm is set up to find a left factor; we therefore take the transpose of H and find the left factor, then the transpose of that factor. However, because of the symmetry of the H we have $H' = H$.

Putting the numbers in and performing an initial factorization yields the left factor

$$H = \begin{pmatrix} \frac{(z-1.470)(z-.680)}{(z-1.075)(z-.930)} & 1 \\ 1 & 1.16 \end{pmatrix}$$

Observe that the outside zero of the numerator in the $(1,1)$ element, which characterize the AR part of the implied process, is larger than thee denominator zero (1.075), so the persistence arising from the MA part of the implied process will be lower than that induced by the AR part. The invertible left factor of H, which is simply the initial candidate factor generated by the Cholesky factorization, is

$$\begin{pmatrix} \frac{.855(z-1.470)}{z-1.075} & 0 \\ \frac{.855(z-.93)}{z-.680} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix}$$

Multiply by a Blaschke factor from the right:

$$\begin{pmatrix} \frac{.855(z-1.470)}{z-1.075} & 0 \\ \frac{.855(z-.93)}{z-.680} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix} \begin{pmatrix} -\frac{z-.680}{1-.680z} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -\frac{.855(z-1.470)}{z-1.075}\frac{z-.680}{1-.680z} & 0 \\ -\frac{.855(z-.93)}{1-.680z} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix}$$

and with cancellation in the upper left element,

$$\begin{pmatrix} \frac{\frac{.855}{.680}(z-.680)}{z-1.075} & 0 \\ \frac{\frac{.855}{.680}(z-.93)}{z-1.47} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix} = \begin{pmatrix} \frac{1.257(z-.680)}{z-1.075} & 0 \\ \frac{1.257(z-.93)}{z-1.47} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix}$$

which has a zero but no poles. The transpose of this, which is the right factor is,

$$\bar{F} = \begin{pmatrix} \frac{1.257(z-.680)}{z-1.075} & \frac{1.257(z-.93)}{z-1.47} \\ 0 & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix}$$

Because of the zero, additional factorization is needed. First, we calculate the constituent elements of the Θ matrix (the coefficient matrix from [2]):

$$A = \begin{pmatrix} .68 \end{pmatrix} \qquad B = \begin{pmatrix} -.884 & -.468 \end{pmatrix} \qquad \Omega = \begin{pmatrix} 1.86 \end{pmatrix}$$

yielding

$$\Theta = \begin{pmatrix} -\frac{.930(z-.494)}{z-1.470} & -\frac{1.022(z-1.0)}{z-1.470} \\ -\frac{1.022(z-1.0)}{z-1.470} & \frac{.459(z-2.025)}{z-1.470} \end{pmatrix}$$

The invertible factor is

$$\Theta^* \bar{F} = \begin{pmatrix} -\frac{.312(z-2.025)}{z-.68} & -\frac{.695(z-1.0)}{z-.68} \\ -\frac{.695(z-1.0)}{z-.68} & \frac{.632(z-.494)}{z-.68} \end{pmatrix} \begin{pmatrix} \frac{1.257(z-.680)}{z-1.075} & \frac{1.257(z-.93)}{z-1.47} \\ 0 & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix} = \begin{pmatrix} \frac{.393(z-2.025)}{z-1.075} & .187 \\ \frac{.874(z-1.0)}{z-1.075} & 1.061 \end{pmatrix}$$

The inverse is

$$\bar{F}^{-1} = \begin{pmatrix} \frac{4.199(z-1.075)}{z-1.075} & -\frac{.739(z-1.075)}{z-2.686} \\ -\frac{3.452(z-1)}{z-2.686} & \frac{1.552(z-2.025)}{z-2.686} \end{pmatrix}$$

We can write this as

$$\bar{F}^{-1} = \begin{pmatrix} \frac{4.199}{z-1.075}\Phi^{-1} & -\frac{.739}{z-2.686}\Phi^{-1} \\ -\frac{3.452(z-1)}{z-2.686} & \frac{1.552(z-2.025)}{z-2.686} \end{pmatrix}$$

Inserting this into (57) yields

$$\begin{pmatrix} B_1 \\ \Theta_1 \end{pmatrix} = \begin{pmatrix} c_2 J & 0 \\ 0 & \Phi^{-1} \end{pmatrix} \begin{pmatrix} \frac{4.199}{z-1.075}\Phi^{-1} & -\frac{.739}{z-2.686}\Phi^{-1} \\ -\frac{3.452(z-1)}{z-2.686} & \frac{1.552(z-2.025)}{z-2.686} \end{pmatrix} \begin{pmatrix} \bar{F}_{11}(r+a)^{-1}\Phi \\ \bar{F}_{12}(r+a)^{-1}\Phi \end{pmatrix} \tag{58}$$

Clearly there will be several cancellations of the Φs. This yields

$$\begin{pmatrix} B_1 \\ \Theta_1 \end{pmatrix} = \begin{pmatrix} c_2 J & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{4.199}{z-1.075} & -\frac{.739}{z-2.686} \\ -\frac{3.452(z-1)}{z-2.686} & \frac{1.552(z-2.025)}{z-2.686} \end{pmatrix} \begin{pmatrix} \bar{F}_{11}(r+a)^{-1} \\ \bar{F}_{12}(r+a)^{-1} \end{pmatrix} \tag{59}$$

Remembering that $\bar{F}_{11}(r+a)^{-1}$ and $\bar{F}_{12}(r+a)^{-1}$ are constants, it is evident that Θ_1 (the obfuscation coeffient) is not a constant:

$$\Theta_1 = -\bar{F}_{11}^{-1}(r+a)\frac{3.452(z-1)}{z-2.686} + \bar{F}_{12}^{-1}(r+a)\frac{1.552(z-2.025)}{z-2.686}$$

which is an ARMA(1,1) transfer function. Thus, the large shareholder adds persistence (mainly through the AR part) to the fundamental process.

We can be more explicit about the solution using the annihilator lemma (using discrete time):

$$\bar{F}^{*-1}\Big|_{z=a} = \begin{pmatrix} \frac{4.199(.93-1.075)}{.93-1.075} & -\frac{3.452(.93-1)}{.93-2.686} \\ -\frac{.739(.93-1.075)}{.93-2.686} & \frac{1.552(.93-2.025)}{.93-2.686} \end{pmatrix} = \begin{pmatrix} .347 & -.138 \\ -.0611 & .967 \end{pmatrix}$$

Our equation is then

$$\Theta_1 = -.347\frac{3.452(z-1)}{z-2.686} - .138\frac{1.552(z-2.025)}{z-2.686} = -\frac{1.41(z-1.155)}{z-2.686}$$

which is an ARMA(1,1) filter. Moreover, it does not cancel with Φ, and so the effective fundamental process will be

$$(1-\Theta_1)\Phi = \frac{2.410(z-1.790)}{z-2.686}\Phi.$$

A slight variant of this example is discussed in the main text. [Figure 1 is for a slightly different case: the lower right element of the matrix to be factored is 1.1 rather than 1.18.] As the adjustment cost is increased (not shown), the amplification filter flattens out and shrinks.

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