Risk-sharing and contagion in networks

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Abstract

The aim of this paper is to investigate how the capacity of an economic system to absorb shocks depends on the specific pattern of interconnections established among financial firms. The key trade-off at work is between the risk-sharing gains enjoyed by firms when they become more interconnected and the large-scale costs resulting from an increased risk exposure. We focus on two dimensions of the network structure: the size of the (disjoint) components into which the network is divided, and the "relative *density*" of connections within each component. We find that when the distribution of the shocks displays "fat" tails extreme segmentation is optimal, while minimal segmentation and high density are optimal when the distribution exhibits "thin" tails. For other, less regular distributions intermediate degrees of segmentation and sparser connections are also optimal. We also find that there is typically a conflict between efficiency and pairwise stability, due to a "size externality" that is not internalized by firms who belong to components that have reached an individually optimal size.

1 Introduction

Recent economic events have made it clear that looking at financial entities in isolation gives an incomplete, and possibly very misleading, impression of the potential impact of shocks to the financial system. In the words of Acharya *et al.* (2010) "current financial regulations, such as Basel I and Basel II, are designed to limit each institution's risk seen in isolation; they are not sufficiently focused on systemic risk even though systemic risk is often the rationale provided for such regulation." The aim of this paper is precisely to investigate how the capacity of the system to absorb shocks depends on the pattern of interconnections established among financial firms, say banks.

More specifically, we intend to study the extent to which the risk-sharing benefits to firms of becoming more highly interconnected (which provides some insurance against relatively small shocks) may be offset by the large-scale costs resulting from an increased risk exposure (which, for large shocks, could entail a large wave of induced bankruptcies). That is, we want to analyze the trade-off between risk-sharing and contagion. Clearly, this trade-off must be at the center of any regulatory efforts of the financial world that takes a truly systemic view of the problem. This paper highlights some of the considerations that should play a key role in this endeavor. In particular, by formulating the problem in a stylized and analytically tractable framework, it examines how the segmentation of the system into separate components, the density of the connections within each component as well as asymmetries in the pattern of connections should be tailored to the underlying shock structure. It also sheds light on the key issue of whether the normative prescriptions on the optimal pattern of linkages are consistent with the individual incentives to form or remove links.

We analyze a model in which there is a network consisting of N nodes, each of them interpreted as a firm. For simplicity, in most of the paper we shall consider the case where all firms are *ex ante* identical and are endowed with the same level of assets and liabilities. But *ex post* they will be different since we assume that, with some probability, a shock hits a randomly selected firm. The first *direct* effect of such a shock is to decrease the income generated by the firm's assets, thus possibly leading to the default of the firm if its resulting income falls short of its liabilities. But if this firm has links to other firms, the latter will also be affected. To be specific, let us think of the presence of a link between two firms as reflecting an exchange of the assets they are endowed with. Then, the overall network of connections generates patterns of mutual exposure between any pair of directly or indirectly connected firms, the magnitude of such exposure decreasing with their respective network distance as well as on the degree (the number of links) of each firm. Thus, when a shock hits a firm, any other firm in the same network component[1] becomes affected in proportion to its exposure to that firm and has to default when its share of the shock exceeds the value of the firm's assets, net of its liabilities. So, in the end, it is the overall 'network' structure that determines how any given shock affects different firms and what is its overall aggregate impact on the whole system.

In order to concentrate the analysis on our basic trade-off – insurance versus contagion – we focus most of the analysis on two dimensions of the network structure. One is the size of the (disjoint) components into which the network is divided, i.e. the degree of *segmentation* of the system. The other concerns the "relative *density*" of connections within each component, as measured by the average network distance between a firm and any other firm in a component.

Network density is important because, as explained, different network distances yield different degrees of exposure. In this respect, our analysis will largely focus on contrasting two polar (symmetric) cases: (i) completely connected components, where there is a direct

[1] As usual, a component of the network is defined as a maximal set of nodes (i.e. firms) that are directly or indirectly connected.

link between any pair of firms in each component; (ii) minimally connected components, where each firm has the minimum number of links (i.e. two) required to obtain indirect connectivity to every other firm in the component, firms are then arranged in a ring. In the first case (complete components), the mutual exposure between any pair of firms in the same component is exactly the same. Instead, in the second case (minimally connected components), the reciprocal exposure between two firms is heterogeneous, falling with their network distance.

A key objective is then to identify the architecture of the system that minimizes the expected number of defaults in the system, that is the best solution of the trade-off between risk sharing and contagion. This involves finding both the optimal degree of segmentation as well as the optimal link density within each component.

The maximum extent of risk sharing clearly obtains when all firms belong to a single and fully connected network. But this configuration exhibits the widest exposure of firms in the system to shocks and so a large shock, which would affect all firms in the system, could lead to extensive default. There are two alternative (and in some cases complementary) ways of reducing such exposure. One is by segmentation, which isolates the firms in each component from the shocks that hit any other component. The second one is by reducing the density of connections in each component, which buffers the network-mediated propagation along this component of any shock that hits one of its firms.

The paper proposes a stylized model that captures the essence of the problem and allows to study the aforementioned trade-off under a fairly general structure of the shocks. A first set of our results can be summarized as follows. We find that when the probability distribution of the shocks exhibits "fat tails" (i.e. attributes a high mass to large shocks), the optimal configuration involves a maximum degree of segmentation – that is, components should be of the minimum possible size. This reflects a situation where the priority is to minimize contagion. Instead in the opposite case, where the probability distribution places high enough mass on relatively small shocks, the best configuration has all firms arranged in a single component. The main aim in this latter case is to achieve the highest level of risk sharing. These two polar cases, however, do not exhaust all possibilities. For we also find that for other, more complex specifications of the shock structure (e.g. mixtures of fat and thin tails) intermediate arrangements are optimal, i.e. the optimal degree of segmentation involves medium-sized components.

It is interesting to note that all of the previous conclusions hold irrespectively of whether components are of either of the two network structures considered, i.e. completely or minimally connected structures. But, in fact, an analogous trade-off between risk-sharing and contagion can be attained by varying the network density. This is explored by our second set of results. The potential advantage of a minimally connected structure is that firm exposure between firms in a component is not uniform but decays with network distance. Thus, if shocks are fairly large, only a fraction of the firms will default while all would default if they were completely connected. An immediate consequence of this observation is that a minimally connected structure always exhibits an optimal degree of segmentation that is lower (or, equivalently, a component size that is larger) than a completely connected one. More importantly, we show that while for the shock distributions mentioned in the previous paragraph a completely connected structure is superior to all minimally connected ones, for other, less regular shock distributions (for instance those with a high mass on a small range of large shocks as well as on a wide range of small and medium shocks) a minimally connected structure is optimal, better than any segmentation in complete components. In these cases low density is preferred to more segmentation as the mechanism for limiting contagion when the shocks are large.

The results described so far refer to environments where all firms have the same size and face the same shock distribution. We also explore an extension with asymmetric firms, which makes it natural to study asymmetric structures. We first analyze asymmetries arising from the fact that the shocks hitting different firms may be distributed differently. We show that in this case the optimal arrangements require assortative matching, that is, components formed by firms all with the same shock distribution. Size asymmetries, on the other hand, turn out to be unimportant for optimal group formation, if the size of the returns for larger firms is scaled in the same way as the probability of the shocks hitting them.

Finally, the paper addresses the issue of whether the requirements for the optimality of the network structure of the system are compatible with the incentives of individual firms to establish links. Formally, we analyze this issue by examining the *Coalition-Proof Equilibria* (CPE) of a network formation game, where any group (i.e. coalition) of firms can jointly deviate, to avoid the multiplicity issues associated to the coordination problems present in these games. We find that there is typically a conflict between efficiency and individual incentives. This conflict derives from the fact that CPE typically exhibit asymmetries in component size and these are inconsistent with efficiency. There are, therefore, positive externalities associated to symmetry in component size that are not internalized at a CPE which, in general, exhibits some firms lying in a component that is inefficiently small.

To sum up, our analysis highlights that the efficient configuration of the pattern of connections among financial firms – concerning, in particular, its segmentation, density, and the handling of size asymmetries – crucially depends on the nature of the shocks faced by the system. Since, as explained, one cannot generally expect that social and individual incentives be aligned, an important role for policy opens up. Our model, of course, is too stylized to allow for the formulation of concrete policy advice. It provides, however, a theoretical framework that is useful to understand the core issues and trade-offs involved, laying then the basis for the derivation of more specific policy implications.

We end this introduction with a brief review of the related literature. The research on financial contagion and systemic risk is quite diverse and also fast-growing. Hence we shall provide here only a brief summary of some of the more closely related papers.[2]

Allen and Gale (2000) pioneered the study of the stability of interconnected financial systems. They analyze a model in the Diamond and Dybvig (1983) tradition, where a single, completely connected component is *always* the efficient network structure, i.e. the one that minimizes the extent of default. Our model, in contrast, shows that a richer shock structure can generate a genuine trade-off between risk-sharing and contagion and that in some circumstances a certain degree of segmentation and/or low density may turn out to be efficient.

Freixas *et al.* (2000) consider an environment where a *lower density* of interaction, even though it limits risk sharing, has the positive consequence of reducing the incentives for deposit withdrawal. A positive role for *segmentation* is then obtained by Leitner (2005) in a model where, unlike ours, no role is played by the mode of interaction, since within a component risk is always shared completely. Allen, Babus, and Carletti (2011) consider a six-firm environment where each firm needs funds for its investment. Since these investments are risky, firms may gain from risk diversification, achieved by exchanging shares with other firms. This gives rise to a financial network for which two possibilities are considered: a segmented and an unsegmented structure. The paper then analyses the different effects in these two structures of the arrival of a signal indicating that *some* firm in the system will have to default.

There is then a complementary line of literature that, in contrast with the papers just mentioned, studies the issue of contagion and systemic risk in the context of large networks (typically, randomly generated). In many of these papers, the approach is numerical, based on large-scale simulations (see e.g. Nier *et al.* (2007)). In this line, the recent paper by Blume *et al.* (2011) integrates the mathematical theory of random networks with the strategic analysis of network formation, to study the question of what is the socially optimal degree (i.e., connectivity) of the system and whether this is consistent with agents' incentives to disconnet. They find that social optimality is attained around the threshold where a large component emerges, but individuals will want to connect beyond this point. This induces, as in our paper, a conflict between social and individual optimality, which in their case is due to the fact that agents do not internalize the effect on others of adding new channels (i.e. links) for the spread of contagion.

Finally, we should mention a large empirical and policy-oriented literature, a large part of which aims to devise summary measures of the network of inter-firm (mostly banks) relationships able to predict systemic failures. For example, Battiston *et al.* (2012) propose a

[2] The reader is referred to Allen and Babus (2009) for a recent survey on how risk sharing in financial contexts can lead, through contagion, to large systemic effects. There is also a large body of literature that studies the general problem of risk sharing in non financial contexts, largely motivated by its application to consumption sharing in poor economies that lack formal insurance mechanisms. Paradigmatic examples are the papers by Bramoullé and Kranton (2007), Bloch *et al.* (2008), and Ambrus *et al.* (2011).

measure of centrality (*Debtrank*, inspired on the *Pagerank* centrality measures that Google uses to rank webpages) which takes into account the feedback effect that failures can have on neighbors at different distances, and estimates its value for US data. Similarly, Denbee *et al.* (2011) propose a measure of centrality á la Katz-Bonacich, following Ballester, Calvó-Armengol and Zenou (2006), and apply it to English data. Of particular interest in this respect is Elsinger, Lehar and Summer (2011) who, using Austrian data, show that correlation in banks' asset portfolios is the main source of systemic risk. This is interesting, because portfolio correlation is precisely the driver of failure risk in our model.

The rest of the paper is organized as follows. Section 2 presents the model and its continuum approximation which proves quite convenient. Section 3 characterizes optimal financial structures for various properties of the shock distribution. Section 4 analyzes the network formation process and shows the tension between strategic stability and optimality. Finally, Section 6 considers some extensions. For convenience, all formal proofs of our results are relegated to an Appendix.

2 The Model

2.1 The Environment

We consider an environment with N ex ante identical, risk-neutral financial firms and a continuum of small investors. At any given point in time, each firm has an investment opportunity - a project - which requires an initial payment $I = 1$ and yields a random gross return $\widetilde{R}$ at the end of the period. The resources needed to undertake the project are obtained by issuing liabilities (e.g. deposits or bonds) on which a deterministic rate of return must be paid.

The gross return of the project is random as with some probability q the firm is hit by a negative shock. If no shock hits, the return equals some normal level R. The shock can in turn be small, which we label as s, or large (b), with conditional probabilities π_s and π_b, respectively. When a small shock hits, the firm experiences a loss of some fixed size L_s, so its gross return is $R - L_s$. When the shock is large, the loss $\widetilde{L}_b$ is *ex ante* random, with support $[L_s, \infty)$ and distribution function $\Phi(L_b)$. Summarizing, the gross return on a firm's project is:

$$\widetilde{R} = \begin{cases} R \text{ with prob. } 1 - q \\ R - L_s \text{ with prob. } q\,\pi_s \\ R - \widetilde{L}_b \text{ with prob. } q\,\pi_b \end{cases} \tag{1}$$

Since the return on a firm's investment is subject to shocks, while the return promised to its creditors is deterministic, when the firm is hit by a shock it may be unable to meet the required payments on its liabilities, in which case it must *default*. Default entails two types of costs. First, there are the liquidations costs – for simplicity, we assume that these costs leave no resources available to make any payment to creditors at the time of default.

In addition, there are additional costs deriving from the fact that a defaulting firm stops operating and hence loses any future earnings possibility. These costs are assumed to be substantial, so that the value of a firm is maximized when its probability of default at any point in time is minimized.

There is then a large set of investors, who are the source of the supply of funds to firms. Investors are risk neutral and require an expected gross rate of return equal to r in order to lend their funds in any given period. Since firms may default, in which case creditors receive a payment equal to zero, the nominal gross rate of return M on the deposits to the firms must be greater or equal than r. Specifically, if we denote by φ the *ex ante* probability that any given firm defaults (an endogenous variable), we must have:

$$M = \frac{r}{1-\varphi}. \tag{2}$$

For the risk of default to be an issue, we assume:

A1. (i) $R(1-q) > r$
(ii) $R - L_s \leq r$.

The first inequality ensures that a firm's project is viable, that is, its expected return when it is not hit by a shock exceeds the expected return which must be paid to lenders. The second inequality implies, since $r \leq M$ (and this inequality is strict as long as $\varphi > 0$), $\widetilde{L}_b \geq L_s$, that if a firm can only draw on the revenue generated by its project it is surely unable to pay depositors (and hence must default) whenever a shock (whether s or b) hits its return.

Since, as stated above, default entails a significant cost for a firm, a firm may benefit from entering *risk sharing arrangements* with other firms which allow it to diversify risks. Here we consider the case where these arrangements take the form of swaps of assets between firms, that is, exchanges of claims to the yields of the firms' investments, prior to the realization of the uncertainty (similarly to Allen, Babus and Carletti (2011)). The (possibly iterative) procedure through which each firm exchanges shares on its whole array of asset holdings can be viewed as a *securitization* process of the firms' claims.

More precisely, let us posit that each firm exchanges a fraction $1-\theta$ of its standing shares, giving rights to the return on its investments, for shares held by other firms. Such an exchange entitles the firm to a fraction of the returns on the other firms' investments. Note that, due to the *ex ante* symmetry of firms, this exchange takes place on a one for one basis, as it involves assets of equal expected return. The specific pattern of exchanges among firms is formalized by a network, where a direct linkage between two firms reflects the fact that they undertake a *direct* exchange of their assets. We allow for these asset swaps to occur repeatedly. *Indirect* connections are then also formed, whereby a firm ends up having claims on the returns of projects of firms who swapped assets with the firms it trades with, and so on. As a consequence a pair of firms lying at a certain distance in the network will have some reciprocal exposure to the yields of each other's projects provided the number of exchange rounds is high enough – in particular, as high as their network distance.

As a result of this process of asset exchanges, the return on a firm's assets becomes a weighted average of the yield of its own project and the yields of the projects of the firms it traded with, directly or indirectly. A convenient way of representing a network structure is through a matrix A of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \tag{3}$$

where for each i, j, $(i \neq j)$, $a_{ij} \geq 0$ denotes the fraction of shares in the investment project run by firm i that is owned by firm j. By construction, the following adding-up constraints must then hold:

$$\sum_{j=i}^{N} a_{ij} = 1 \qquad\qquad i = 1, 2...., N \tag{4}$$

In addition, given that all firms are ex-ante symmetric it is natural to focus our attention on the case where the pattern of asset exchanges is also symmetric across firms. This, together with the fact that all portfolio swaps are conducted on a one-for-one basis imply that A is symmetric, i.e.

$$a_{ij} = a_{ji} \quad \text{for all} \quad i, j = 1, 2, ..., N.$$

In what follows we shall compare different network configurations in terms of social welfare. From the assumptions that default costs are significant, all agents are risk neutral and firms are *ex ante* identical, it follows that social welfare is maximized when the expected number of defaults of firms in the system is minimized.[3] It is then easy to see that this criterion is equivalent to that of minimizing the individual probability that any single firm defaults.[4] In this case, therefore, social and individual objectives are aligned from an *ex ante* viewpoint.

[3]Beale et al. (2011) propose a similar criterion to evaluate and compare different financial systems, based on the minimization of a "systemic cost function" defined as the expectation of a convex function of the number of defaults in the system. Thus, in this case, not only the expected number of defaults matters, but also its variability.

[4]To see this, note that the *ex ante* probability of default of an individual firm φ is equal to $\sum_m \rho(m)\varphi(m)$, where $\rho(m)$ stands for the probability that m firms default and $\varphi(m)$ for the conditional probability that any particular firm defaults when there are a total of m defaults in the system. Then, since

$$\varphi(m) = \frac{1}{N} + (1 - \frac{1}{N})\frac{1}{N-1} + \cdots + (1 - \frac{1}{N} - \cdots - \frac{1}{N})\frac{1}{N-m+1} = \frac{m}{N}$$

we obtain that the expected number of defaults in the system is $\sum_m \rho(m)\, m = \varphi N$, i.e. is proportional to the individual default probability.

We shall assume that all network structures involve the same "externalization of risk" by each firm. That is each firm, after all rounds of asset exchanges have been completed, holds a claim to a fraction α of the yield of its own project, and the residual fraction $1-\alpha$ of claims to other firms' projects. Formally, we have $a_{ii} = \alpha$ for each $i = 1, .., N$ and every network configuration. The value of α is taken to be a parameter of the model, greater or equal than 1/2.

Considerations of moral hazard provide a natural motivation for the presence of a lower bound on α. If in fact to operate one's project some costly effort needs to be exerted, a way to induce a high enough effort level is to make sure that the firm retains some minimum share[5] of the project's returns.[6]. On the other hand, an upper bound on the fraction of claims retained to its own project is explained by the benefits of risk sharing and the fact that, as we will see below, a situation with all firms in isolation is never optimal.

Alternative structures, therefore, differ only in terms of how the fraction $1-\alpha$ of the risk that is externalized is distributed among the rest of the firms. This allows a more meaningful comparison across different network structures.

We assume that shocks are rare and thus each period at most one firm is hit by a shock. Given (1), this can be motivated by postulating that, even if shocks hit firms in a stochastically independent manner, the probability q that a shock hits any given firm is so low that the probability that two or more shocks arrive in a single period is of an order of magnitude that can be ignored.[7]

When a shock of size L hits the return on the project run by some firm i, the exposure to it of all firms in the system is given by Ae_iL, where e_i is the i-th unit vector $[0, .., 1, ..0]^\top$. Hence firm i defaults in response to such a shock when

$$\alpha(R-L) + \sum_{j \neq i} a_{ij} R < M, \quad \text{that is,} \quad \alpha L > R - M,$$

while firm $k \neq i$ defaults whenever

$$\left(\alpha + \sum_{i \neq j \neq k} a_{kj} \right) R + a_{ki}(R-L) < M.$$

We readily see from the above expressions that when a firm exchanges its assets with other firms, the firm reduces its exposure to the shocks hitting its own project but at the same

[5]This requirement is analogous, for example, to a well-known provision in the recent Dodd-Frank act, recently passed in the USA to regulate further the financial system, by which, under certain circumstances, "a securitizer is required to retain not less than 5 percent of the credit risk..." (see http://www.sec.gov/about/laws/wallstreetreform-cpa.pdf).

[6]In particular, the lower bound of 1/2 ensures that a firm always retains a larger share of claims on its project than any other firm, even in the case where the firm only exchanges assets with one other firm.

[7]Or, as an extreme formalization of this idea, we could model time continuously and assume that the arrivals of small and big shocks to each firm are governed by independent Poisson processes with fixed rates π_s and π_b, respectively.

time it becomes exposed to the shocks affecting the projects of those firms with which the firm in question is directly or indirectly connected.

On the nature of those shocks, we make the following key assumption:

A2. (i) $\pi_s > N\pi_b$.
(ii) $\alpha(R - L_s) + (1-\alpha)R \geq \frac{r}{1-Nq\pi_b}$.

Part (i) of the above assumption says that small (s) shocks are significantly more likely than big (b) ones. In particular, it is more likely that a given firm is hit directly by a small shock than a big shock hits any of the firms. Part (ii) then ensures that, by exchanging a fraction $1-\alpha$ of its shares with shares of other firms, no firm defaults when a s shock hits any firm in the environment. The term on the right-hand side of A2(ii) constitutes in fact an upper bound on the gross rate of return M on deposits when no default occurs with an s shock, since the probability of default φ of a firm is no larger than $Nq\pi_b$, that is the probability that a b shock hits anywhere in the system. On the other hand, the term on the left-hand side of the inequality in A2(ii) constitutes the gross return of a firm that swapped a fraction $1-\alpha$ of its shares in the event where the firm is hit by a s shock. Since $\alpha \geq 1/2$ this term is also an upper bound on the firm's revenue when any other firm in the system is hit by a s shock.

Thus, the inequality in A2(ii) indeed guarantees that exchange and securitization, that is link formation, allows firms to fully insure against the s shocks. At the same time, diversification also exposes a firm to the risk of contagion when a b shock hits any of the firms directly or indirectly linked to the firm. However, recalling that Assumptions A1(ii) and A2(i) imply that a firm in isolation always defaults when hit by a s shock and that s shocks are much more likely than b shocks, it follows that the probability of default of a firm is always lower when it exchanges assets with other firms than when it is in autarky.

While any pattern of asset trades with degree of externalization of the own shock given by α allows firms to attain full insurance against s shocks, the default performance in the event of b shocks will typically not be the same across different financial network structures. In general, that is, these structures will be markedly different in terms of the extent of contagion they induce when big shocks hit the system. Hence, one of the primary aims of this paper is to understand what configurations minimize those detrimental side effects of risk-sharing.

REMARK 1 *We should point out that in the environment considered the mutual exposure between firms comes from the cross-ownership of their shares, not from mutual lending relationships. Hence the default of one firm has no direct implication for the solvency of other firms, the possibility of contagion of a large shock hitting a firm only comes from the correlation of their portfolio returns.*

2.2 Financial Structures

We shall consider financial structures that differ along two dimensions: segmentation and network density. For the moment, we shall also restrict our analysis to symmetric configurations – which means, in particular, that all components of the network are of identical size and have the same network density. This, however, implies no loss of generality for our normative analysis since, as we will show, the welfare maximizing configurations are essentially symmetric. In contrast, in Section 6, where the implications of strategic stability are examined, some asymmetries in the components' sizes will emerge.

By *segmentation* we mean the partition of the N firms into disjoint components. Each component is formed by firms that are either directly or indirectly linked by the exchange of assets (and hence the crossownership of shares), while there is no trade across components. The measure of the segmentation is given by the number C of equal-sized components in which the set of firms is divided, or equivalently by the number $K \equiv \frac{N}{C} - 1$ of other firms to whom every firm is linked[8]. In terms of the matrix A, this amounts to having a block diagonal structure with C blocks along the diagonal. The larger the segmentation (the smaller K), the fewer the firms affected by a given shock but, *ceteris paribus*, the larger their mutual exposure and hence the probability of default if a b shock hits them. At the two extremes of segmentation, we have the case $K = N - 1$, where all firms are connected (directly or indirectly), and $K = 1$, where each firm only engages in trade with a single other firm. We shall allow for all possible values of K, between 1 and $N - 1$ and explore their welfare implications.

By *network density* we refer to the proportion of direct versus indirect linkages within each component. As we said in the previous section, a firm is directly linked to another one if the two are involved in a direct exchange of assets. Instead, two firms are only indirectly linked if they are connected by a multiple-link path in the financial network. In this case, they end up holding claims to each other's projects if there is a repeated exchange of assets (or, say, repeated rounds of securitization). In general, we can have different levels of density, ranging from 1 (all linkages are direct) to the case where the amount of direct connectivity is minimal. For simplicity, our analysis will just compare these two extremes, which are described in more detail in what follows.

For the choice of the financial structure to be non trivial, we assume that the number of firms in the system is not too small, that is:

A3. $N > 4$.

Minimally Connected Structures In this case firms display the minimum number of links required to be connected, directly or indirectly (i.e. through a path of some length), to all other firms in the component. In a symmetric component, it must then be that all

[8]We ignore for now all possible integer problems regarding these magnitudes.

firms have exactly two links and the architecture is that of a ring. Letting K denote, as explained above, the degree of segmentation of the system, the size of each component is $K+1$. The pattern of (direct) exchanges of assets among the firms in this component is thus described by the $(K+1)\times(K+1)$ matrix (for some suitable labeling of the firms in it)

$$B_K = \begin{pmatrix} \theta & (1-\theta)/2 & 0 & \cdots & 0 & (1-\theta)/2 \\ (1-\theta)/2 & \theta & (1-\theta)/2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (1-\theta)/2 & 0 & 0 & \cdots & (1-\theta)/2 & \theta \end{pmatrix},$$

that is each firm exchanges a fraction $(1-\theta)$ of its assets with its two trading partners and retains the rest for itself. These trades are then iterated m times in a repeated process of securitization. In the second round each firm trades the composite asset given by claims on its project and of those of its two neighbors as obtained in the first round. By exchanging these assets the firm acquires claims on the projects of firms that are at network distance two, that is neighbors of its neighbors. And so on. The pattern of exposures among firms in the component at the end of this process is described by the matrix A_K, obtained by repeated composition of the matrix B_K with itself m times, i.e.

$$A_K = (B_K)^m. \tag{5}$$

We posit that $m = K/2$. This is the minimal value of m that ensures that any pair of firms that are either directly or indirectly linked with each other end up with a nonzero share in each other's projects, that is, that all entries of A_K are stricly positive. Thus all firms in the component bear some reciprocal exposure. In addition, it is immediate to see that this exposure falls with the network distance among any two firms in the ring (i.e., the entries of A_K are such that $a_{ij} > a_{lq}$ if and only if $|i-j| < |l-q|$).[9] Finally, θ is set at a level such that, as required in the general specification in the previous section, each firm retains a fraction α of claims on its own project; that is, all the main-diagonal entries of the matrix A_K are equal to α.

Complete Structures This corresponds to the situation where all firms in each component are directly linked among them – so, in the language of networks, all components are completely connected. The exchanges of assets among firms within the component is captured by a square matrix A_K of the form:

[9] It is immediate to see that the same properties hold for any other $m \geq K/2$. As $m \to \infty$, the matrix A_K displays, in the limit, all off-diagonal entries with the same value, i.e. the same property, as we see below, exhibited by complete structures.

$$A_K = \begin{pmatrix} \alpha & (1-\alpha)/K & \cdots & (1-\alpha)/K \\ (1-\alpha)/K & \alpha & \cdots & (1-\alpha)/K \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha)/K & (1-\alpha)/K & \cdots & \alpha \end{pmatrix}, \tag{6}$$

where α is the fraction of its own assets a firm keeps for itself. Evidently in this case the pattern of exposures as described by the above matrix would not change if the exchanges were iterated and hence we can set $m = 1$ with no loss of generality.

In the next sections we will compare financial structures differing in the two above dimensions in terms of their ability to minimize the expected number of defaults when shocks hit. We will identify the optimal degree of segmentation (that is the optimal size of the components) as well as the optimal network density (that is, whether it is preferable to have a ring or a completely connected structure).

Let $\varphi(A; \Phi(.))$ denote the probability of bankruptcy of an arbitrary firm i when the financial network structure is described by the matrix A and the distribution of the b shocks is given by $\Phi(.)$. We have so:

$$\begin{gathered} \varphi(A; \Phi(.)) = \\ q\pi_b \Pr\{\alpha(R - L_b) + (1-\alpha) R) < M\} + q\pi_b \textstyle\sum_{j \neq i}^{N} \Pr\{R - M < a_{ij} L_b\} \end{gathered} \tag{7}$$

where M satisfies (2). The first term on the right hand side is the probability of default when a b shock hits the firm under consideration, the second term is the probability of default of the same firm when a b shock hits any of the other $N-1$ firms. By the symmetry of A the probability of default is the same and given by the above expression for all i.

As argued before[10], the expected number of defaults in the system is $\varphi(A; \Phi(.)) N$. Hence the optimal financial structure is given by the value of K and the network density for which $\varphi(A; \Phi(.))$ is minimal.

It is interesting to note that the first term on the second line of (7) is the same for all the financial structures considered, as it does not depend on A. That is, the probability of default of a firm whose project is hit by a b shock is the same across all these structures. They only differ for their ability to limit contagion, that is to prevent large b shocks from generating the default of the firms connected to the one directly hit by the shock. We will see that even though there is no cost of forming linkages the optimal structure may display fewer connections that in the one with a single, completely connected component, and this is a key feature where we depart from other work as for instance Allen Gale (2001) and Allen, Babus, Carletti (2011).

[10] See footnote 4.

The Continuum Approximation

The pattern of risk exposure induced by either the complete or the ring structures can be graphically depicted through a function that, for each firm i, specifies the fraction of the claims to the yield of the project of firm i that is held by any other firm j in the component, as a function of the (integer) "network distance" between the two firms. This corresponds to the values of the terms on a row of the matrix A_K. Given the assumption that $\alpha \geq 1/2$ this function always reaches the highest value when $i = j$. In the case of the complete structure, this function takes then a constant value for all other $j \neq i$, while for the ring it is a step function, monotonically decreasing.

Because of the discreteness of the domain of this function – in particular, for the case of the ring structure - and the integer problems faced in the determination of the optimal degree of segmentation, a formal analysis of the firms' probability of default for different network configurations becomes quite involved and tedious to carry out. To make the analysis more tractable, we study in what follows a continuum approximation of our model, and in particular of the function above, that abstracts from these considerations and still allows to capture the essential features of the problem[11] and to approximate the value of $\varphi(A; \Phi(.))$.

In the continuum approximation of the model, N is taken to be the measure of the firms in the system and the same applies to $K + 1$, now the measure of firms belonging to a certain component. The returns on a firm's project are the same as in (1). To keep a formal parallelism with the discrete formulation, when a shock occurs, it directly hits a unit measure of firms in one component. These firms, therefore, play the role of the single firm directly hit by a shock in the discrete context.

Segmentation is modeled as in the previous section, except that the size of a component, $K + 1$, need not be an integer and can now be any real positive number lying between 1 and N. Differences in network density, on the other hand, are modeled as follows. If the component is complete, the pattern of exposure to the returns on other firms' projects is as in the discrete model: the exposure to a shock that hits any other firm in the component[12] is constant and equal to $(1 - \alpha)/K$.

On the other hand, if the component has a ring structure, the pattern of risk exposure is described now by the following continuous function $f(d; K)$, where $d \in [0, K/2]$ is the

[11] This continuum formulation can be seen as representing a limit description of a context consisting of a large number of firms, each of them is of small size. Our preferred motivation, however, is to view it simply as a smooth approximation of a context where the number of firms is discrete but not necessarily large.

[12] More precisely, that hits any other firm in the mass K of firms in the component that is not directly hit by a shock when the firm under consideration is hit.

ring distance to the set of firms directly hit by the shock:

$$\begin{aligned} f(d;K) &= \alpha - \tfrac{\alpha-H}{H} d && \text{for } 0 \leq d \leq H \\ &= \tfrac{HK}{K-2H} - \tfrac{2H}{K-2H} d && \text{for } H < d < K/2 \\ &= 0 && \text{for } d = K/2. \end{aligned} \tag{8}$$

The function is defined for $K \geq 1$ and the value of H is determined by the following adding-up constraint:

$$2\int_0^{K/2} f(x;K)\mathrm{d}x = (\alpha - H)\,H + 2H^2 + H\,(K/2 - H) = 1-\alpha, \tag{9}$$

that is:

$$H = \frac{2(1-\alpha)}{K+2\alpha}. \tag{10}$$

Recalling that $\alpha \geq 1/2$, $K \geq 1$, it is immediate to verify that the function $f(d;K)$ in (8) exhibits the following properties that are satisfied by the exposure function in the discrete set-up:

$$f(d;K) \text{ is positive and decreasing for all } K/2 > d > 0, \tag{11}$$

$$f(0) = \alpha, \tag{12}$$

$$f(K/2) = 0, \tag{13}$$

as well as (9). First, (11) says that every firm in a component is affected by a shock hitting any other firm in the component, but with an intensity that decreases with the distance to the source of the shock. In addition, (12) states that the level of exposure to a firm at minimal distance is the same as that to a direct shock, while (13) says that the exposure becomes vanishing small when the distance to the source of the shock is maximal in the component. Finally, (9) captures the requirement that each firm externalizes a fraction $1-\alpha$ of its risk.

Note that $f(d;K)$ is a two-piece linear function with the kink at the bisectrix (i.e. at a distance $d = H$ such that $f(H) = H$) and is concave or convex depending on whether, respectively, K is smaller or larger than $2(1-\alpha)/\alpha$. An illustration of how it approximates the original function for the discrete setup is displayed in Figure 1.

Include former Figure 1 HERE

Using the above specification of the function describing the pattern of exposure for the ring and the complete structures and all possible levels of K, we are now in a position to determine the extent of default induced by any given shock of magnitude L for every possible financial structure. Note first that whether the unit mass of adjacent firms directly hit by a shock default or not is independent, as already argued in the previous section, of the underlying financial network structure. For they will default if, and only if,

$$L > \frac{R-M}{\alpha}.$$

If this inequality is violated and the firms hit by the shock do not default, no other firm in the corresponding component defaults either. This simply follows from the fact that $\alpha \geq (1-\alpha)/K$ since $\alpha \geq 1/2$, $K \geq 1$ and $f(d; K) \leq \alpha$ for any $d \geq 0$. But if those firms directly affected by the shock do default, what happens to all the others in the component naturally depends on the size $K+1$ of the component and on the pattern of the connections within it.

In the case where the interaction pattern is complete, the uniformity of the exposure has the following immediate implication: all firms indirectly affected (i.e. not hit by the shock but lying in the component affected) will also default if, and only if

$$L > \frac{K}{1-\alpha}(R-M) \tag{14}$$

whereas none of those firms will default otherwise. Thus, if we let $g_c(L; K)$ stand for the mass of firms that default when the shock hits some other firm in their component, that magnitude is given by the following step function:

$$g_c(L;K) = \begin{cases} 0 & \text{if } L \leq \frac{K}{1-\alpha}(R-M) \\ K & \text{if } L > \frac{K}{1-\alpha}(R-M) \end{cases} \tag{15}$$

In contrast, when the component is connected through a ring interaction structure (as captured by the exposure pattern $f(\cdot)$), the conclusion is, in general, not so dichotomic. For, in this case, whether a firm in the component defaults or not depends on its ring distance d to those firms that have been directly affected. It defaults if, and only if,

$$L > \frac{1}{f(d;K)}(R-M).$$

which is to be contrasted with (14). Hence the threshold that marks the relevant "default range" is given by the distance $\hat{d}$ such that

$$f(\hat{d}; K) = \frac{R-M}{L}, \tag{16}$$

so that a firm defaults if, and only if, its distance d from the set of firms directly hit by the shock is such that $d < \hat{d}$. Under a ring structure, therefore, the effect of different levels of the shock on the mass of firms defaulting is not discontinuous as in the complete structure. Rather, as the magnitude L of the shock increases, the mass of firms defaulting among those indirectly affected by it grows gradually, as determined by the function $g_r(L;K) \equiv 2f^{-1}((R-M)/L;K)$. This function is easily seen to have the following form *(see an illustration in Figure 2 for* $K = 20$ *and* $\alpha = 1/2$ MOVE THIS TO FIGURE'S CAPTION*)*:

$$g_r(L;K) = \begin{cases} 0 & \text{for } L \leq \frac{R-M}{\alpha} \\ \frac{2\alpha H}{\alpha - H} - \frac{2H}{\alpha - H}\frac{R-M}{L} & \text{for } \frac{R-M}{\alpha} \leq L \leq \frac{R-M}{H} \\ K - \frac{K-2H}{H}\frac{R-M}{L} & \text{for } L \geq \frac{R-M}{H} \end{cases}. \tag{17}$$

Include former Figure 2 HERE

3 Optimal Financial Structures

In this section we identify the optimal network segmentation and network density, that is, the financial structure that minimizes the expected mass of firms defaulting, for the continuum approximation of the model. To this end, it is convenient to use the functions specified in (17), and (15) indicating the mass of firms defaulting in a component for the ring and the complete structures for all possible realizations of L.

Formally, the optimal degree of segmentation is obtained as a solution of the following optimization problem:

$$\begin{array}{c} \min_{K_i,C} \sum_{i=1}^{C} \frac{K_i+1}{N} \mathbb{E} g_\nu(\widetilde{L}_b; K_i) \\ \text{s.t.} \sum_{i=1}^{C} \frac{K_i+1}{N} = 1, \end{array} \tag{18}$$

respectively for the ring ($\nu = r$) and the complete ($\nu = c$) structures. The objective function is the expected mass of firms not directly hit by a shock which default when a b shock occurs. As explained in the previous section, minimizing this function also minimizes the expected mass of firms defaulting in the system. The choice variables are not only the number C of different components in the system, but also the size K_i+1 of each component i, $i = 1, .., C$. We allow here for the possibility of asymmetric structures, where the system is divided into components of different size. It will be shown, however, that as we already said the solution is (essentially) symmetric for all cases under consideration, with $K_i = K$ for all i. This implies that the problem (18) can be reduced to minimizing $\mathbb{E} g_v(\widetilde{L}_b; K)$ with respect to K.

By comparing then the optimum values of the solution of (18) for $v = r$ and $v = c$ we obtain the optimal network density.

Note that the value of $\mathbb{E} g_v(\widetilde{L}_b; K)$ depends not only on the distribution $\Phi(L_b)$ but also on the other parameters of the model, α, R, r, L_s. The specific values of these parameters have however little interesting bearing on the analysis and our primary focus is on the effects of the shock distribution on the optimal network structure. Hence in what follows we set[13] $\alpha = 1/2$, normalize $R - r$ to unity and set also $L_s = 1$. Furthermore, even though M is an endogenous variable, determined by (2), and varies with the underlying network structure, for the purpose of determining the optimal structure we can without loss of generality ignore this and replace M with the constant value r (and hence $R - M = 1$). This follows from the fact that the ordering of structures, in terms of their default probability, that is obtained when $R - M$ is set equal to one, is not going to change if the value of M is adjusted - and increased - to reflect the probability of default.[14]

[13] A convenient consequence of this choice of α is that, just as for the discrete version of the model, the pattern of exposure is exactly the same for the complete and the ring structure if the component size is the smallest (i.e. $K = 1$). The difference between the two structures then grows wider the higher is K.

[14] This only underestimate the actual default probability for all structures.

Proposition 1 *Let two firm structures A_1 and A_2, and let $qN\pi_b \equiv \varepsilon$ and $D(A_2; q\pi_b, M_{A_2}(q\pi_b)) \equiv \varphi(A_2; q\pi_b, , M_{A_2}(q\pi_b))/q\pi_b$. There exists some $\overline{\varepsilon}$ small enough, such that for $D(A_2; 0, M_{A_2}(0)) > D(A_1; 0, M_{A_1}(0))$, then $D(A_2; \varepsilon, M_{A_2}(\varepsilon)) > D(A_1; \varepsilon, M_{A_1}(\varepsilon))$ for all $\varepsilon < \overline{\varepsilon}$.*

Proof of Proposition 1 Rewriting equation (2) we get

$$M_{A_i}(\varepsilon)\left[1 - \varepsilon\left(\frac{1}{N}\Pi + \frac{N-1}{N}\frac{D(A_i; \varepsilon, M_{A_i}(\varepsilon))}{N-1}\right)\right] = r$$

where Π is the probability that a firm defaults when directly hit by a b shock, the same for all structures since α is also the same across structures. The function $D(A_i; \varepsilon, M_{A_i}(\varepsilon))$ can be assumed continuously differentiable since it is just a (possibly complicated) polynomial of ε. Therefore, one can also claim that the function $M(\varepsilon)$ defined by equation (2) is continuously differentiable, at least locally (invoking the implicit function theorem). Then, the derivative of M evaluated at $\varepsilon = 0$ is

$$\left.\frac{\partial M_{A_i}(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = M_{A_i}(0) = r$$

Then we can rely on a Taylor expansion to write:

$$M_{A_2}(\varepsilon) - M_{A_1}(\varepsilon) \geq [D(A_2; 0, r) - D(A_1; 0, r)]\varepsilon - \Delta(\varepsilon)$$

where $\Delta(\varepsilon)$ is the residual function. By assumption, $D(A_2; 0, r) - D(A_1; 0, r) > 0$. Therefore, since $\Delta(\varepsilon)$converges to zero when $\varepsilon \to 0$ at the rate of ε^2, we can assert that there exists some $\overline{\varepsilon} > 0$ such that for all positive ε such that $\varepsilon < \overline{\varepsilon}$., $M_{A_2}(\varepsilon) - M_{A_1}(\varepsilon) > 0$. ■

We organize the analysis in three parts. First, in Subsection 3.1 we identify some clear-cut conditions regarding the probability distribution of the b shocks under which the optimal segmentation is one of the two polar extremes – maximal or minimal – and the optimal degree of connectivity is complete. Then, in Subsection 3.2 we extend the analysis to more general specifications of the shock distribution, for which intermediate levels of segmentation are optimal. Finally, in Subsection 3.3 we identify scenarios where the optimal structure exhibits not only intermediate levels of segmentation but also a low density of connections, as embodied by the ring structure.

3.1 Polarized segmentation

In order to get a clear understanding of the forces at work, we shall start by examining the case where the probability distribution of the b shocks is of the Pareto family with support $[1, \infty)$ and density $\gamma/L_b^{\gamma+1}$. By modulating the decay parameter γ, this formulation already allows the discussion of many questions of interest such as the contrast between fat and thin tails in the shock distribution (i.e. between scenarios where large shocks are relatively frequent or not). As mentioned above, our analysis will be carried in two steps. Firstly, we shall characterize how γ affects the optimal degree of segmentation (as described by K) for the ring and for the complete structure. Secondly, we shall compare these two structures.

Let $D_r(K,\gamma) = \mathbb{E}_\gamma g_r(\widetilde{L}_b; K)$, i.e. the expected mass of firms in a ring of size $K+1$ who default when indirectly hit by a b shock with a Pareto distribution with parameter γ. We have so[15]:

$$\begin{gathered} D_r(K,\gamma) = \int_1^\infty g_r(L;K)\, \mathrm{d}\Phi(L;\gamma) \\ = \int_{\frac{R-M}{H}}^\infty \left(K - \frac{K-2H}{H}\frac{1}{L}\right)\frac{\gamma}{L^{\gamma+1}}\mathrm{d}L + \int_{\frac{R-M}{\alpha}}^{\frac{R-M}{H}} \left(\frac{2\alpha H}{\alpha-H} - \frac{2H}{\alpha-H}\frac{1}{L}\right)\frac{\gamma}{L^{\gamma+1}}\mathrm{d}L \\ = \gamma\left[K\frac{1}{\gamma(K+1)^\gamma} - [K(K+1)-2]\frac{1}{(\gamma+1)(K+1)^{\gamma+1}}\right] + \\ 2\gamma\left[-\frac{1}{K-1}\frac{1}{\gamma(K+1)^\gamma} + \frac{2}{K-1}\frac{1}{(\gamma+1)(K+1)^{\gamma+1}} + \frac{1}{K-1}\frac{1}{\gamma 2^\gamma} - \frac{2}{K-1}\frac{1}{(\gamma+1)2^{\gamma+1}}\right] \end{gathered} \tag{19}$$

We study next how the above expression behaves as K varies in its admissible range (recall that the minimum[16] admissible value of K is 1 and its maximal value is $N-1$). We establish in the following proposition that whenever $\gamma > 1$ (i.e. the distribution function of $\widetilde{L}_b$ does not have fat tails) $D_r(K,\gamma)$ attains a minimum at the highest value of K, $N-1$. On the other hand, when $\gamma < 1$ (i.e. the distribution has fat tails) the function attains a minimum at $K = 1$.

Lemma 1 *When the shock $\widetilde{L}_b$ has a Pareto distribution, the component size which minimizes the mass of firms defaulting is minimal ($K = 1$) if $\gamma > 1$, and maximal ($K = N-1$) if $\gamma < 1$.*

On this basis, we can easily determine the optimal segmentation pattern in the system. To this end we must however also take into account the constraint present in (18), $\sum_{i=1}^{C}(K_i + 1)/N = 1$. Given the findings of the above lemma, this constraint only binds when N is odd and $\gamma < 1$, as in such case a structure with all components of the optimal (minimal) size 2 is not feasible. To find the optimal structure in this case we establish the concavity of $D_r(K,\gamma)$:

*As indicated, however (see Footnote **??**), the gist of the result extends naturally to cases where N is any large real number, in which case the maximum segmentation involves $N/2$ components of size 2 and a residual. This conclusion readily follows from the following lemma.*

Lemma 2 *When $\gamma < 1$ the function $D_r(1/2, K, \gamma)$ is strictly concave in K, for all $K > 1$.*

From Lemmas 1 and 2 it immediately follows that the optimal structure when $\gamma < 1$ and N is odd is 'almost' symmetric, with $\frac{N}{2} - 1$ components of minimal size 2 and a residual component of size 3. In all other cases, the optimal ring structure is the one with all components of the same optimal size, determined in Lemma 1. We can then summarize our findings in the following:

[15] In the second equality below we also used the parameter values for α, $R-M$ specified at the end of the previous section, which imply that $H = 1/(K+1)$.

[16] Strictly speaking, the expression in (19) holds for $K > 1$. For $K = 1$ we have $D_r(1,\gamma) = \gamma\left[K\frac{1}{\gamma(K+1)^\gamma} - [K(K+1)-2]\frac{1}{(\gamma+1)(K+1)^{\gamma+1}}\right]$. However, since we show in the proof of Lemma 1 that $D_r(K,\gamma)$ is continuous at $K = 1$, in what follows it suffices to work with (19).

PROPOSITION 1 *Suppose the shock $\widetilde{L}_b$ has a Pareto distribution. With $\gamma > 1$ (no fat tails) the optimal degree of segmentation for the ring structure is minimal, with a single component with N firms. Otherwise, with $\gamma < 1$ segmentation is maximal, with all components of the minimal size 2 (except one, of size 3, when N is odd).*

The previous result shows that there is indeed a trade-off between risk sharing and contagion. On the one hand, when the distribution of the shocks exhibits fat tails (hence large shocks are relatively likely), the predominant consideration is to control contagion rather than achieve risk sharing. This leads to minimizing the expected number of defaults by breaking the network into disjoint components of minimal size, which limits the extent to which a shock may spread its consequences far into the system. Instead, when the distribution has no fat tails, the most important consideration becomes risk-sharing, which is maximized by placing all firms in a single component.

Next, we turn to studying the analogous question for the case where the components are completely connected. In this case, the expected mass of firms who default in a completely connected component of size $K+1$ when indirectly hit by a b shock is:

$$D_c(K,\gamma) = \mathbb{E}_\gamma g_c(\widetilde{L}_b; K) = K \ \Pr\left(L \geq 2K\right) = K \ \left(\frac{1}{2K}\right)^\gamma. \tag{20}$$

Hence

$$\frac{\partial D_c}{\partial K} = -\left(\gamma - 1\right)\left(\frac{1}{2K}\right)^\gamma \gtrless 0 \quad \Longleftrightarrow \quad \gamma \lessgtr 1,$$

which readily implies that the optimal component size is again minimal when $\gamma < 1$ (the shock distribution has fat tails), while it is maximal in the case where $\gamma > 1$. Hence the optimal segmentation structure is the same as for the ring, with all components of the optimal size, when the constraint $\sum_{i=1}^{C}(K_i+1)/N = 1$ does not bind. In contrast, when $\gamma < 1$ and N odd, and hence this constraint binds, the optimal structure is now exactly symmetric, with $\frac{N}{2} - 1$ components, all of the same size (slightly larger than 2), since we show that $D_c(\cdot)$ is a convex function of K. Hence we have the following:

PROPOSITION 2 *When the shock $\widetilde{L}_b$ has a Pareto distribution, the optimal degree of segmentation for the completely connected structure is minimal (one single component) if $\gamma > 1$, and maximal, with $N/2$ ($N/2 - 1$ if N is odd) identical components if $\gamma < 1$.*

Finally, we need to compare the optimal ring and the optimal complete network structures in order to identify which of the two is optimal when not only segmentation but also network density can be chosen. In view of Propositions 1 and 2, it is enough to compare the expected mass of firms defaulting under either maximal or minimum segmentation for the ring and the complete structures when, respectively, γ is lower or higher than unity. Note also that when K is at its minimal admissible value, 1, the pattern of exposure is the same for the two structures, $g_c(L_b; 1) = g_r(L_b; 1)$, hence a difference only arises when the optimal component size is greater than 1 for at least one structure. This leads to the following result.

PROPOSITION 3 *If the shock $\widetilde{L}_b$ has a Pareto distribution, for N large enough ($N > 1 + (1+\gamma)^{\frac{1}{\gamma-1}}$), the completely connected structure weakly dominates the ring structure for all values of γ: the two structures are equivalent when $\gamma < 1$ (and N is even and the complete network is uniquely optimal when $\gamma > 1$.*

Combining the results obtained in this subsection, we conclude that if shocks are Pareto distributed, the optimal network always displays maximal density and polarized (maximum or minimum) segmentation. In this case, therefore, all the adjustment to the underlying risk conditions (i.e. to the different values of γ, generating fat or thin tails) is obtained only by varying the segmentation pattern. However, as our subsequent analysis will show, neither the polarized segmentation pattern nor the complete connectivity within components are features maintained for other, more complex, shock distributions.

3.2 Intermediate Degrees of Segmentation

Propositions 1 and 2 establish that, when the distribution of the shocks has a simple Pareto structure and thus it either has, or does not have, fat tails, the optimal degree of segmentation is always extreme, i.e. maximal or minimal. We show next that this is no longer true when the distribution of the shocks is more complex, as for instance when it is given by the mixture of two Pareto distributions.

PROPOSITION 4 *Suppose that the shock $\widetilde{L}_b$ is distributed as a mixture of a Pareto distribution with parameter $\gamma > 1$ and another Pareto distribution with parameter $\gamma' < 1$, with respective weights p and $1-p$. Then, there exist $p_0, p_1, 0 < p_0 < p_1 < 1$, such that, whenever $p \in (p_0, p_1)$, the optimal pattern of segmentation for the completely connected structure is symmetric with components of intermediate size $K^* + 1$, $1 < K^* < N - 1$.*

Abusing slightly previous notation, denote by $D_c(K, \gamma, \gamma', p) = p\mathbb{E}_{\gamma} g_c(\widetilde{L}_b; K) + (1 - p)\mathbb{E}_{\gamma'} g_c(\widetilde{L}_b; K)$ the expected mass of defaults in a complete component of size $K+1$ when a b shock hits some other firm in the component, and this shock distribution is as in the above proposition. We show in the proof that, under the conditions stated in the proposition, the function $D_c(K, \gamma, \gamma', p)$ attains a minimum at an intermediate value $\hat{K} \in (1, N-1)$. A symmetric structure with all components of size $\hat{K}$ is however generically not feasible now (i.e. violates the condition $\sum_{i=1}^{C}(K_i + 1)/N = 1$). We then show that the optimal structure is still symmetric, with all components of size K^*, smaller or equal to $\hat{K}$.

The previous result establishes that an intermediate level of segmentation is optimal for completely connected structures when the shock distribution involves a mixture of Pareto distributions displaying with both fat and thin tails. A similar conclusion arises when the components display a ring structure, although an analytic result in this case is hard to obtain. We illustrate matters, therefore, through the following example.

EXAMPLE 1 *Set* $\gamma = 2$, $\gamma' = 0.5$ *and* $p = 0.95$. *For these values we find that the value of* K *which minimizes* $D_c(K, \gamma, \gamma', p)$ *is* $\hat{K}^c = 5.65$, *and at this value the expected mass of defaults (when a* b *shock hits some other firms in the component) is* 0.13. *The value of* K *which minimizes the corresponding expression for the ring structure,* $D_r(K, \gamma, \gamma', p)$, *is higher,* $\hat{K}^r = 8.02$ *and the expected mass of defaults in this case is also higher, equal to* 0.145. *The fact that* $\hat{K}^r > \hat{K}^c$ *can be heuristically understood as a reflection of the fact that, when arranged optimally, components with a ring structure compensate for their lower density of connections with a larger size. See Figure 1 for a graphical description of the pattern of these two functions.*

To find the optimal financial structure for the whole system we also need to specify the value of N. *Let* $N = 10$. *In this case it is clear that neither a symmetric structure with equal components of size* $\hat{K}^c + 1$ *when completely connected or* $\hat{K}^r + 1$ *when rings is feasible. We find that both for the complete and the ring structures the optimal structure of the system is given by two equal-sized components of size* $K^* + 1 = 5$. *Moreover, in line with the conclusions stated above for the values of expected defaults at* $\hat{K}^r$ *and* $\hat{K}^c$, *we find that the optimal complete structure still dominates the optimal ring structure: the former yields an expected mass of defaults equal to 0.26 in contrast with 0.32 induced by the latter (see Figure 2). This conclusion proves to be robust with respect to other possible specifications of the parameter values of the environment.*

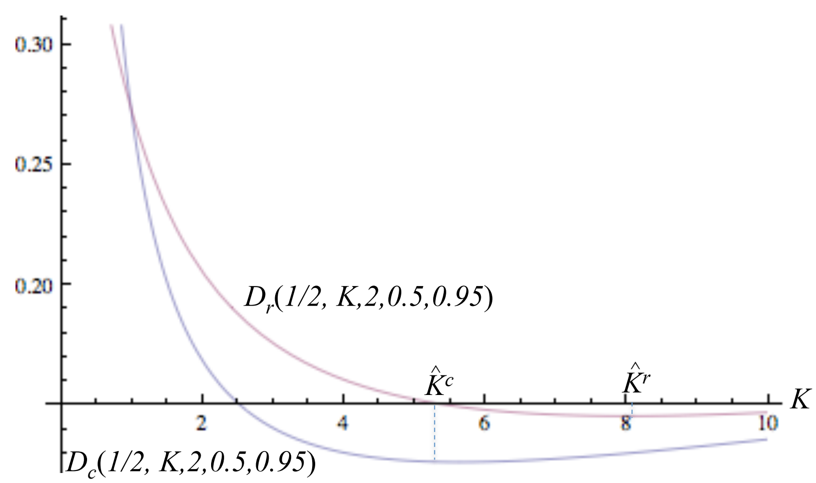

Figure 1: Expected mass of defaults in a given component as a function of its size K for both complete $(\mathbb{E}g_c(\widetilde{L}_b; K))$ and ring structures $(\mathbb{E}g_r(\widetilde{L}_b; K))$.

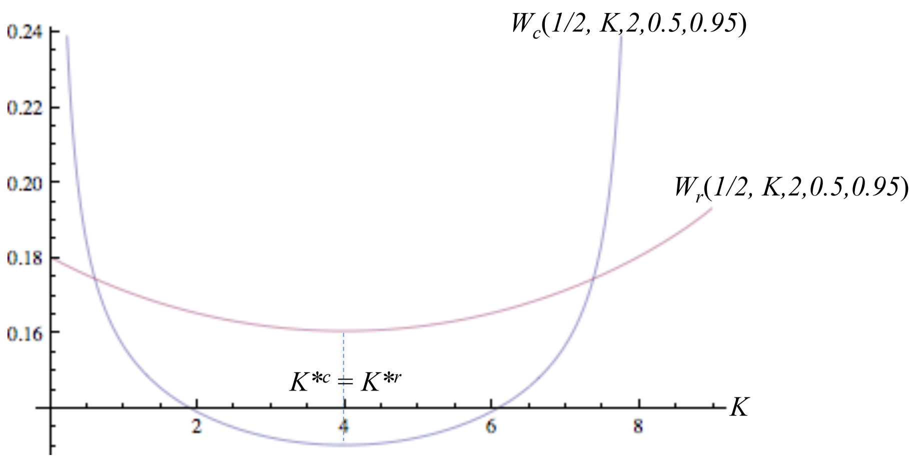

Figure 2: Expected mass of firms that default when indirectly hit by a shock in a system consisting of $N = 10$ firms divided in two components of size K and $N - K$, as a function of K for the case of complete ($D_c(K)$) and ring components ($D_c(K)$).

3.3 Sparse Connections

Let us consider now the case where the probability distribution of the b shocks is not smooth because it has some atoms. More precisely, let $\Phi(L_b)$ be the mixture of a Pareto distribution with $\gamma > 1$ and a Dirac distribution putting all probability mass on a shock of magnitude $\bar{L} > 2(N-1)$. On the one hand, the Pareto distribution considered has no fat tails, which as we saw in Section 3.1 implies that minimal segmentation ($K = N - 1$) is optimal and, in addition, that the completely connected structure dominates the ring structure. On the other hand, the shock $\bar{L}$ induced through the Dirac distribution is such that if such a shock occurs and firms are arranged in a single component, all firms default when they are completely connected while some survive if arranged in a ring. We show below that there is an open region of parameter values for which the second effect prevails over the first one and hence the optimal financial structure is a ring – that is, sparser connections are optimal.

Proposition 5 *Let $\widetilde{L}_b$ be distributed as a mixture of a Pareto distribution with parameter $2 > \gamma > 1$ and a Dirac distribution will all mass on $\bar{L} = 2(N-1)+1$, with weights respectively p and $1-p$. Then, for all values of N such that*

$$N > 1 + \left(\frac{1}{4^{\gamma-1}} - \frac{1}{5^{\gamma}} + \frac{\frac{1}{2^{\gamma-1}} \frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}} \right)^{\frac{1}{2-\gamma}} \tag{21}$$

there exists an open set of values of p such that

$$\frac{(1-p)}{p} < (\gamma - 1) \quad \left(\frac{1}{2(N-1)} \right)^{\gamma}. \tag{22}$$

for which the optimal financial structure is a single ring component.

Condition (22) says that the weight p on the Pareto distribution is sufficiently high so that the optimal segmentation structure is determined by it and a single component is optimal, both for the ring and the complete structures. But, given that there is also a non negligible probability that a large shock arrives that cannot be fully absorbed, some attempt at "controlling the induced damage" may be in order. And this is indeed what the ring achieves – a suitable compromise between the extent of *risk sharing* allowed by extensive *indirect* connectivity (i.e. minimal segmentation) and the limits to wide *risk contagion* resulting from sparse *direct* connections.

4 Stability and optimality

We now examine the relationship between the optimal pattern of linkages derived in the previous section and the individual incentives to form those linkages. We explore, in other words, whether social welfare is aligned with the maximization of individual payoffs. To model the strategic considerations involved in the creation and destruction of links, the network-formation game is assumed to be conducted as follows:

- Firms independently submit their proposals concerning the set of other firms in the whole population[17] N each one of them wants to connect to. Formally, a strategy of each firm i is a subset $S_i \subset N$.
- Links are formed (only) between the firms who reciprocally list each other at the proposal-submission stage. Formally, given a profile of strategies $\mathbf{S} \equiv (S_i)_{i \in N}$ a link between any two firms j and k is established iff $j \in S_k$ and $k \in S_j$.

Given the above network-formation rules, a specific network $\Gamma(\mathbf{S})$ is induced by each strategy profile $\mathbf{S}$. As we said in Section 2.1, the resulting payoff of each firm i is decreasing in $\varphi_i(\Gamma(\mathbf{S}))$, its own default probability resulting from the network induced by $\mathbf{S}$ and is maximal when the default probability is minimal.

In such a network-formation game, an undesirable feature of the standard concept of Nash Equilibrium is that it leads to a vast multiplicity of equilibrium networks, a consequence of the fact that the formation of any link induces a coordination problem between the two agents involved. (As an extreme illustration, note that the empty network can always be supported by a Nash equilibrium where every agent proposes nobody to link with.)

To address this issue, it is common in the literature to consider a strengthening of the Nash equilibrium notion that reduces miscoordination by allowing sets of agents to deviate jointly (see e.g. Goyal and Vega Redondo (2007) or Calvó-Armengol and Ikiliç (2009)).

[17]We use here the same notation used for the mass of firms in the system to denote also the set of firms in the system.

In the framework of the continuum approximation of the model considered here, we shall capture this idea by means of the concept we shall label *Coalition-Proof Equilibrium* (CPE), where any group (i.e. coalition) of agents can coordinate their deviations.

DEFINITION 1 *A strategy profile* $\mathbf{S} \equiv (S_i)_{i \in N}$ *of the network-formation game defines a* Coalition-Proof equilibrium *(CPE) if there is* no subset *of firms* W *of positive measure and a substrategy profile* $(S'_j)_{j \in W}$ *for all of them such that*[18]

$$\forall i \in W, \quad \varphi_i \left[\Gamma \left((S'_j)_{j \in W}, (S_k)_{k \in N \setminus W} \right) \right] < \varphi_i [\Gamma(\mathbf{S})].$$

The CPE notion precludes unilateral profitable deviations by any (positive-measure) set of firms, so it is a obviously a refinement of the standard notion of Nash Equilibrium. It is a very strong refinement in that it embodies no limit on the amount/measure of firms that may coordinate their deviations. But, in general, it would be natural to impose such limit since large-scale deviations may be very difficult to implement. In our case, however, introducing some such limit would have no effect on the results presented below[19], so we choose to avoid it for expositional simplicity. Another requirement typically demanded in Game Theory from coalition-based notions of equilibrium is that the coalitional deviations considered should be robust, in the sense of being themselves immune to a subcoalition profitably deviating from it. Again, this requirement has no bite for our analysis since, we can show[20] that all the profitable deviations that need to be allowed are robust in the aforementioned sense.

Consider first the case where the optimal structure is given by identical components of the optimal size, which minimizes the mass of defaults in a component, that is, where the feasibility constraint $\sum_{i=1}^{C} (K_i + 1)/N = 1$ does not bind (as for instance in Propositions 1 and 2). In that case, it is immediate to verify that the optimal structure is also a CPE. In contrast, we show in what follows that this no longer true when the feasibility constraint binds.

To illustrate this, we analyze the CPE for the class of distributions considered in Section 3.2 and show that there is a conflict between social optimality and individual incentives. Recall that in such setup Proposition 4 established that, for an open set of the parameter space, the optimal configuration involves a symmetric segmentation of the whole population into several (equal-sized) components that are completely connected. Moreover, the size

[18] In line with what was postulated in Section 2, we shall continue to assume that, after any change in connections has been implemented, the agents involved in the change continue to distribute the fraction $1 - \alpha$ of their own assets among all their neighbors (old and new) in a uniform manner

[19] More specifically, for any $\eta > 0$ we could define the notion of η-CPE, such that only joint deviations by at a set of agents of measure no higher than $\eta > 0$ are considered. (The unrestricted notion of CPE obtains if $\eta \geq N$.) We show in the Appendix (see Remark 2) that Proposition 6 remains valid when such notion of equilibrium is considered, for any value of $\eta > 0$.

[20] See again Remark 2 in the Appendix.

$K^* + 1$ of each component is typically smaller than the optimal component size $\hat{K} + 1$, that is the size which minimizes default in a component. Building upon that result, Proposition 6 below shows that in such environment the optimal configuration cannot be obtained as a CPE. This brings about the conflict between individual and social incentives that gives rise to the inefficiency of CPE-configurations. In a CPE displaying complete components the outcome is asymmetric: individual incentives (supported by coalitional deviations) lead to all but one components of individually optimal size $\hat{K} + 1$ and one other component of a size smaller than $K^* + 1$. This is not socially optimal because, in essence, convexity considerations favor a more balanced configuration where the sizes of those components is closer.

PROPOSITION 6 *Consider the same environment of Proposition 4. Then for all $p \in (p_0, p_1)$, the socially optimal configuration cannot be supported at a CPE of the network formation game. Among the completely connected structures, the only CPE configuration is asymmetric with all but one component displaying the size $\hat{K}$ that minimizes $D_c(K, \gamma, \gamma', p_0)$ and one component of a lower size.*

Tha argument proceeds by showing first that in a CPE we cannot have any component of size larger than the individually optimal one $\hat{K} + 1$, because then a subset of firms in that component would benefit by severing some of their linkages. Next, we show that there cannot be two components both of size smaller than $\hat{K} + 1$. In such case a subset of firms in the first component would benefit by severing their linkages with the other firms in that component and forming new linkages with the firms in the second component.

The source of the conflict between social and individual optimality lies in the externality that is imposed by the fact that the feasibility constraint must hold in the aggregate. Hence firms in any component strive for the size that is individually (or component-wise) optimal, which may force other firms to remain in a component that is too small to share risk efficiently. The network-formation game shows how the incentives of each firm to form linkages and engage in risk sharing trades with other firms generate an outcome that is sometimes inefficient. As a simple illustration of the above result, refer back to Example 1, where the efficient configuration involved two completely connected components of common size equal to $K^* + 1 = 5$. We found that the optimal size for each separate component is $\hat{K}^c + 1 = 6.65$, the value that minimizes $D_c(K, 2, 0.5, 0.95)$. In contrast the only CPE configuration with two completely connected components is asymmetric, one displaying a size of 6.65 and the other a smaller "residual size" of 3.35.

The existence of a conflict between efficiency and strategic stability is of course hardly novel nor surprising in the field of social networks (see e.g. Jackson and Wolinsky (1996) for an early instance of it). For, typically, the creation or destruction of any link between two agents must impose externalities on others that are not internalized by the two agents involved in the linking decision. In the context of risk-sharing, this tension has been studied in a recent paper by Bramoullé and Kranton (2007) – hereafter labelled BK – and it is interesting to understand the differences with our approach. We close, therefore, this section with a brief comparison of the two models.

BK consider an environment consisting of a finite number of agents affected by i.i.d. income shocks. Linkages generate risk sharing in a way similar to that of our model, except in two important respects: (a) risk-sharing is complete (i.e. uniform) across all members in a component; (b) there is no risk of contagion, so the size of optimal components is just limited by the fact that links are assumed to be costly. Focusing on the notion of strategic stability (which is weaker than ours),[21] BK are also interested in comparing efficient and equilibrium configurations. They find that whenever equilibrium structures exist (not always), there are at most two asymmetric components, with sizes smaller than the optimal one. The contrast between BK's conclusions and ours derives from the nature of the externality in the two cases. In BK, given that the cost of any new link is borne only by the two agents involved, in equilibrium there is underinvestment in link formation (which is a "public good"). Instead, in our case there are no linking costs and the nature of the externality that is not internalized has to do with the fact that firms who deviate, abandoning a component with the socially optimal size in order to join a component of the individually optimal size, do not consider the effect of their action on the firms with whom links are deleted. Thus, in the end, it is the need to meet an overall feasibility constraint induced by overall population size that typically generates inefficiencies.

5 Asymmetric structures

So far we have concentrated the discussion on a situation where all firms in the system are identical. Although this allows us to obtain analytical results and gather intuition, it is useful to see how our analysis extends when firms are significantly different in size, a situation we often find in the real world.

5.1 Shock and size asymmetry

For this section we always assume the structures are completely connected. To begin with consider that all firms have equal resources in the absence of shocks, but they are different in the distribution of shocks they receive. More precisely, the N firms, are partitioned in subsets $N_1, ...N_n$, so that $N = \cup_{l=1}^{n} N_n$, and for every firm $j \in N_l$ the big shock follows some distribution F_{N_l}. We denote by $D(K, F_{N_l})$ the expected number of defaults, if a big shock hits a firm j belonging to subset N_l in a complete component of size $K+1$. We assume the N firms are organized so that each firm j belongs to a complete component G_i, whose size we denote by K_i so that $N = \cup_{i=1}^{I} G_i$.

[21] Strategic stability allows only for coalitions of at most two players and rules out as well the simultaneous creation and destruction of links. See Jackson and Wolinsky (1996) for details.

The reason why we can write $D(K, F_{N_l})$ just as a function of the size of the component and the distribution of the shock arriving to the individual firm which is hit, and not of the types of shocks arriving to other firms in the component, is that all firms have equal size. When all firms have equal size the proportion they keep of their own projects is α for all of them, and the exposure to shocks of other firms is $(1-\alpha)/K$ [22]. More formally, this implies that the expected number of deaths if $j \in N_l$ is

$$KP\left(\frac{(1-\alpha)L}{K} > 1\right)$$

and this expression only depends on K and the distribution of L.

Define $K^*_{N_l}$ as the minimizer in K of $D(K, F_{N_l})$.

PROPOSITION 7 *Suppose that $|N_l|$ is a multiple of $K^*_{N_l} + 1$ for every l. The optimal configuration of N is such that all firms are in groups with firms of the same type, and of size $K^*_{N_l} + 1$ for firms of type N_l.*

Proof of Proposition 7: The expected number of defaults among firms not directly hit in a component G_i is

$$D_A(G_i) = \frac{K_i + 1}{N} \sum_{l=1}^{n} \frac{|j \in N_l \cap G_i|}{K_i + 1} D(K_i, F_{N_l}).$$

Then the total number of defaults in the system is

$$\begin{aligned}
\sum_{i=1}^{I} D_A(G_i) &= \sum_{i=1}^{I} \frac{K_i + 1}{N} \sum_{l=1}^{n} \frac{|j \in N_l \cap G_i|}{K_i + 1} D(K_i, F_{N_l}). \\
&= \sum_{i=1}^{I} \sum_{l=1}^{n} \frac{|j \in N_l \cap G_i|}{N} D(K_i, F_{N_l}).
\end{aligned}$$

But exchanging the summation indices, we get that

$$\begin{aligned}
\sum_{i=1}^{I} D_A(G_i) &= \sum_{l=1}^{n} \sum_{i=1}^{I} \frac{|j \in N_l \cap G_i|}{N} D(K_i, F_{N_l}) \\
&\leq \sum_{l=1}^{n} \sum_{i=1}^{I} \frac{|j \in N_l \cap G_i|}{N} D\left(K^*_{N_l}, F_{N_l}\right) \\
&= \sum_{l=1}^{n} D\left(K^*_{N_l}, F_{N_l}\right) \sum_{G_i=1}^{I} \frac{|j \in N_l \cap G_i|}{N} \\
&= \sum_{l=1}^{n} \frac{|N_l|}{N} D\left(K^*_{N_l}, F_{N_l}\right)
\end{aligned}$$

[22] In particular this means that even though the risks are different, the exchange of assets is still one for one.

but since $|N_l|$ is a multiple of $K^*_{N_l}+1$

$$\sum_{l=1}^{n}\frac{|N_l|}{N}D\left(K^*_{N_l},F_{N_l}\right)$$

is the expected number of defaults if every firm is part of a group of optimal size $K^*_{N_l}+1$ with firms of the same type.■

We now assume some firms j have a different size β[23] meaning that the return on firm j's project when no shock affects it is βR and at the same time the probability of a shock of size βL is equal to the probability of a shock of size L for a firm of size 1. In order to maintain equal values in the trade, a firm of size β exchanges assets with a firm of size 1 in a proportion $1/\beta$ to 1. We keep the same structure as before in terms of the distribution of shocks, so that we denote by N_l^1 is the set of firms of type l and size 1, and N_l^β is the set of firms of type l and size β. Let a complete component G_i of firms with different sizes, say K_1 firms of size 1 and K_β firms of size β, and where each firm $j\in N_l$. Let $K=K_1+\beta K_\beta$. Then we claim that the expected deaths in a component of size K when a firm j is hit whose shocks are of type N_l is $D\left(K,F_{N_l}\right)$ so that it is independent of the composition of shock types and sizes for the rest of the group. This is so because this implies that the expected number of deaths if the firm which is hit $j\in N_l$ is size 1 is

$$K_1P\left(\frac{(1-\alpha)L}{K}>1\right)+\beta K_\beta P\left(\frac{\beta(1-\alpha)L}{K}>\beta\right)=KP\left(\frac{(1-\alpha)L}{K}>1\right)\tag{23}$$

if the firm which is hit $j\in N_l$ is size β the expected number of deaths is

$$K_1P\left(\frac{(1-\alpha)\beta L}{\beta K}>1\right)+\beta K_\beta P\left(\frac{(1-\alpha)\beta L}{K}>\beta\right)=KP\left(\frac{(1-\alpha)L}{K}>1\right)\tag{24}$$

Notice that the two expressiones (23) and (24) are identical and only depend on K and the distribution of L as claimed.

Then

Define $K^*_{N_l}$ as the minimizer in K of $D\left(K,F_{N_l}\right)$. The proof of the following proposition now follows immediately from the one of Proposition 7

PROPOSITION 8 *Suppose that $\left|N_l^1\right|+\beta\left|N_l^\beta\right|$ is a multiple of $K^*_{N_l}+1$ for every l. The optimal configuration of N is such that all firms are in groups with firms of the same type l, and of size $K^*_{N_l}+1$.*

[23]The analysis is basically identical with a larger variety of sizes but the notation would be more cumbersome, so we restrict the presentation to firms of two sizes.

5.2 Asymmetry in structure

For simplicity, we consider here a situation with two types of firms. One type involves firms identical to the ones we have been considering so far (whose size we can normalize to unity). The other type involves firms of a larger size $\beta > 1$. Such a larger size has two implications. First, the returns on these firms' project when no shock affects them is βR, that is, scaled up by β compared to the smaller firms, and, naturally, the same factor applies to the value of their liabilities. Second, they face a probability of being directly hit by a shock that is also β times larger (βq), while the size distribution of this shock is equal to that of small firms (L_s and $\widetilde{L}_b$). In a sense, one can view a large firm as analogous to a completely connected component of smaller firms of the sort we have been considering so far. The only key difference is that, since they all belong to the same entity, there is no longer the requirement that each "unit" of the composite group should remain responsible of a proportion α of its own assets and corresponding liabilities.

Having different kinds of firms forces us to examine new types of structures in which large and small firms may play asymmetric roles. The key new question is whether it is optimal that firms engage in risk sharing trades with other firms of the same size, or rather with firms of a different size. To this end, our analysis will focus on comparing two structures: (a) the first one, *symmetric*, where components are completely connected, and each component involves only firms of identical size; (b) the second one, asymmetric, where components have a *"star" structure*, consisting of a large firm that acts as a central hub, directly connected to various small firms.

In this framework, there is no real gain in using the continuum approximation of the pattern of exposure among firms in the two structures considered. Hence we revert to the original specification, where each firm is conceived as a discrete entity (which can now be small or large, of size one or β). More precisely, we consider the case where there are only two large firms and 2β small ones. Hence in the type (a) structure we have two completely connected components, of the same total size: one with the two large firms, the other with the 2β small firms. In contrast, in the type (b) structure we have two identical star components, each consisting of β small firms that are solely connected to a large firm (that is, the large firm acts as a hub and there are β spokes of unit size). Hence in this case a component features heterogenous firms, of different sizes, in it.

The first step is to specify the pattern of exchange in the presence of firms of different sizes and hence to determine the induced risk exposure in each of the two cases. In case (a), the situation is analogous to that of the completely connected structures considered in Subsection 2.2. The exposure pattern can then be described by a matrix A_K of the form specified in (6) for a component of size $K + 1$, with $K = 1$ (resp. $K = 2\beta - 1$) for the complete component consisting of large (resp. small) firms.

On the other hand, in a star component (case (b)) the pattern of mutual exposure obtained after the direct exchanges of assets is:[24]

$$\widetilde{B} = \begin{pmatrix} \theta & (1-\theta)/\beta & (1-\theta)/\beta & \cdots & (1-\theta)/\beta \\ (1-\theta) & \theta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\theta) & 0 & 0 & \cdots & \theta \end{pmatrix}$$

The entries of the matrix $\widetilde{B}$ reflect the fact the exchange of assets among firms of different size is no longer one for one: a large firm (indexed by $i = 1$) must offer only a share $(1-\theta)/\beta$ of its assets for a larger share $(1-\theta)$ in the assets of small firms (indexed by $i = 2, 3, ..., \beta + 1$). Maintaining the feature that trades are iterated a minimal number of times so that each firm has a nonzero exposure to any other firm in the component, we need here one round of securitization so that the pattern of exposure among firms after both direct and indirect trades is $\widetilde{A} = \widetilde{B}^2$, or

$$\widetilde{A} = \begin{pmatrix} \theta^2 + (1-\theta)^2 & 2\theta(1-\theta)/\beta & (1-\theta)/\beta & \cdots & (1-\theta)/\beta \\ 2\theta(1-\theta) & \theta^2 + (1-\theta)^2/\beta & (1-\theta)^2/\beta & \cdots & (1-\theta)^2/\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\theta(1-\theta) & (1-\theta)^2/\beta & (1-\theta)^2/\beta & \cdots & \theta^2 + (1-\theta)^2/\beta \end{pmatrix}.$$

As in Subsection 2.2, the value of θ is set so that each firm retains a fraction at least $\alpha = 1/2$ of claims to the yields of its own project. Hence from the above expression of $\widetilde{A}$ we have:

$$\alpha = \theta^2 + (1-\theta)^2/\beta, \tag{25}$$

from which we see that the value of θ satisfying (25) is monotonically decreasing in β. Hence only the small firms will be able to attain the maximal degree of risk "externalization" that is feasible $(1-\alpha)$. Instead, large firms, whose assets are worth β more than those of small ones, cannot possibly attain the same level of externalization via repeated asset exchanges with smaller firms and are forced to retain a larger share

$$\alpha' = \theta^2 + (1-\theta)^2 > \alpha \tag{26}$$

on their own project; the difference $\alpha' - \alpha$ is larger the larger is β.

As in Section 2.1, a firm i defaults when a shock L hits the yield of the project of firm j (possibly $i = j$) in the same component when the firm's exposure to the shock, $\tilde{a}_{ij}L$, exceeds the firm's net returns (where $\tilde{a}_{ij}$ denotes the ij entry of $\widetilde{A}$). This amounts now to a different condition if firm is big or small: if i is small, default occurs when $\tilde{a}_{ij}L > 1$, while if it is large the condition is $\tilde{a}_{ij}L > \beta$.

We show in the next result how the two structures fare, in terms of expected number of defaults, for different values L of the magnitude of the b shock which hits a randomly selected firm:

[24] We dispense here for notational simplicity with any reference to the component size ($K = \beta + 1$).

Proposition 9 *For $\beta > 2$, expected defaults are lower in the symmetric than in the star structure whenever*

$$\frac{1}{1-\alpha'} < L \leq 2\beta, \tag{27}$$

while they are lower in the star structure whenever

$$2(2\beta - 1) < L \leq \max\left\{\frac{\beta - 1}{\alpha' - 1/2}, \frac{\beta^2}{1-\alpha'}\right\}. \tag{28}$$

The result is illustrated in Figure 3.

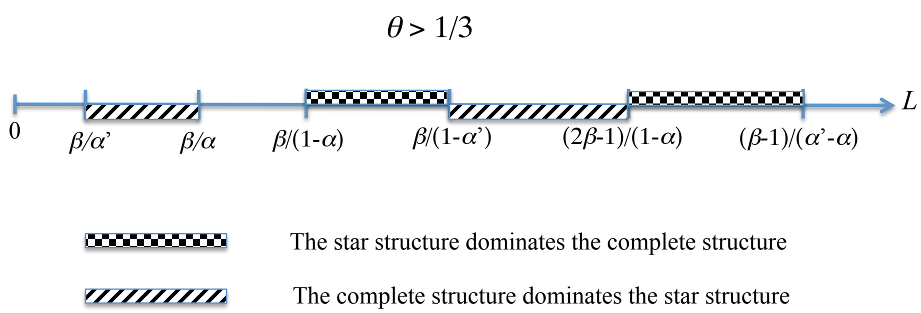

Figure 3: The dark (resp. light) shaded areas are the the magnitudes L of the b shock for which the star (resp. symmetric) structure dominates (i.e. yields a lower expected number of defaults). The non-shaded areas are the values of L for which both structures yield the same expected defaults.

First, the superiority of the symmetric, completely connected structure in the range given by (27) is easy to explain. In the star structure a large firm engages in risk sharing trades with a set of smaller firms. This ends up limiting the possibilities of risk sharing when a shock hits a large firm, also because the large firm is forced to hold more of its own assets than in the symmetric structure, to match the lower value of the assets of the smaller firms located on the spokes. As a consequence, when a shock of an intermediate size as in (27) hits a large firm, it triggers the default of the small firms in the star structure, as well as possibly of the large firm, while the same shock can be absorbed with no defaults in a symmetric structure, in which a large firm shares risk with the other large firm.

On the other hand, it is precisely this limitation to the risk sharing possibilities in a star structure that protects the hub as well as the other spokes against the shocks hitting a small firm. As a consequence, the star structure performs better than the symmetric structure for shocks of larger size, as in (28). In a symmetric component, these shocks are so large that they trigger the default of all the firms linked to a firm directly hit by the shock. In contrast, in a star component none of these firms defaults when the shock hits one of the spoke firms, only the small firm hit defaults. The large firm acts a sort of buffer, preventing a sizable fraction of the shock to spread and cause further defaults in the component (in contrast with the situation just described for the symmetric component).

6 Conclusion

We have proposed a stylized model to study the problem that arises when firms need to share resources to weather shocks that can threaten their survival, but then are exposed to the risk coming from those same connections that help them in the time of need. Depending on the characteristics of the shock distribution, a wide variety of different possibilities can be optimal. For example, maximal segmentation in small groups is optimal if big shocks are likely, while very large groups are optimal when most shocks are of moderate magnitude. There are also conditions, however, when an intermediate group size is optimal or when groups should be large but display some internal "detachment" (i.e. sparse connectivity).

The former consideration pertain to social optimality, i.e. to the minimization of defaults. We have explored whether such overall objective is aligned with individual optimality. And we have seen that, in general, there is a conflict between strategic incentives and social welfare. This tension arises from the fact that when a component attains the size that minimizes the default probability of their members, it will block admitting new members from a smaller components, ignoring the negative externality of their behavior. Finally, we have also studied asymmetric structures and found that certain asymmetries (e.g. a "central" agent acting as a hub) can have useful properties as a *firebreak* in certain cases.

There are many issues that this paper did not study in depth. Although we have identified conditions under which sparse internal connections and asymmetries are beneficial, we do not have a theorem providing general conditions under which alternative topologies are optimal. Since our work has highlighted a disparity between efficiency and equilibrium outcomes, it would be important to extend our normative analysis in this manner. Only then would it be possible to understand better what are the options and consequences of alternative policy measures impinging on the incentives of agents (say, banks) to connect for the purpose of sharing risk. Another important extension would be to integrate such risk-management decisions with other considerations (e.g. cooperation, exploitation of synergies) that also underlie economic connections in the real world.

Appendix

Proof of Lemma 1: Rearranging terms in (19) and simplifying we get:

$$D_r(K,\gamma) = \left(K\left(\frac{1}{\gamma+1}\right) - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma} + \frac{1}{K-1}\frac{1}{(\gamma+1)}\left(\frac{1}{2}\right)^{\gamma-1}, \quad (29)$$

and hence

$$\frac{\partial D_r}{\partial K}(K,\gamma) = -\frac{1}{(K-1)^2}\frac{1}{(\gamma+1)}\left(\frac{1}{2}\right)^{\gamma-1} + \left(K\left(\frac{-\gamma}{\gamma+1}\right)\frac{1}{K+1} + \frac{2}{K-1}\frac{1}{K+1}\frac{\gamma}{\gamma+1} + \left(\frac{1}{\gamma+1}\right) + \frac{2}{(K-1)^2}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}. \tag{30}$$

Now note that the inequality $\partial D_r(K,\gamma)/\partial K > 0$ is equivalent to:

$$\frac{2\,(K-1)}{K+1}\gamma + (K-1)^2 + 2 > (K+1)\left(\frac{K+1}{2}\right)^{\gamma-1} + \gamma K\frac{(K-1)^2}{K+1},$$

or

$$(K-1)^2\left(1-\frac{\gamma K}{K+1}\right) + 2\left(1+\gamma\frac{K-1}{K+1}\right) > (K+1)^{\gamma}\,\frac{1}{2^{\gamma-1}},$$

which can be rewritten as

$$(K-1)^2 + 2 + \gamma\,(K-1)\,(2-K) > (K+1)^{\gamma}\,\frac{1}{2^{\gamma-1}}. \tag{31}$$

So, using the identities

$$(K-1)^2 + 2 + (K-1)\,(2-K) = K+1$$

and

$$(K+1)^{\gamma}\,\frac{1}{2^{\gamma-1}} = 2\left(\frac{K+1}{2}\right)^{\gamma}$$

we can equivalently write (31) as follows:

$$\frac{K+1}{2} - \left(\frac{K+1}{2}\right)^{\gamma} - \frac{1}{2}(1-\gamma)\,(K-1)\,(2-K) > 0.$$

Denote by $\Xi(K,\gamma)$ the term on the left hand side of the previous inequality, conceived as a function of K and γ. Then, to complete the proof, we establish the following property:

$$\forall K > 1, \quad \Xi(K,\gamma) \gtrless 0 \Leftrightarrow \gamma \lessgtr 1. \tag{32}$$

To show this property, note first that $\Xi(K,1) = 0$ for all K, so that $\frac{\partial D_r}{\partial K}(K,\gamma) = 0$ for $\gamma = 1$ and all K. On the other hand,

$$\begin{aligned}\frac{\partial \Xi}{\partial \gamma}(K,\gamma) &= -\left(\frac{K+1}{2}\right)^{\gamma}\ln\frac{K+1}{2} + \frac{1}{2}\,(K-1)\,(2-K) \\ &\leq -\ln\frac{K+1}{2} + \frac{1}{2}\,(K-1)\,(2-K)\,,\end{aligned}$$

the inequality being strict for all $K > 1$. It is then easy to verify that the terms on the two sides of the above inequality are equal to 0 when $K = 1$ and the term on the right hand side is negative[25] for all $K > 1$, establishing (32) and hence also $\partial D_r(K,\gamma)/\partial K \gtrless 0 \Leftrightarrow \gamma \lessgtr 1$.

[25]We have in fact

$$\frac{d\left(-\ln\frac{K+1}{2} + \frac{1}{2}\,(K-1)\,(2-K)\right)}{dK} = \frac{K+1-2K^2}{K+1} < \frac{2K\,(1-K)}{K+1} < 0$$

We conclude, as stated in the proposition, that the minimum of $D_r(K,\gamma)$ is attained at the maximum admissible value of K (i.e. $N-1$) when $\gamma > 1$, while it is attained at the lowest value of K (i.e. $K=1$)[26] when $\gamma < 1$. This completes the proof of the proposition. ■

Proof of Lemma 2 Note first that the expression of $\partial D_r(K,\gamma)/\partial K$ obtained in (29) can be conveniently rewritten as follows:

$$\begin{aligned}
&\tfrac{\partial D_r}{\partial K}(K,\gamma) = -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\
&+\left(\tfrac{-K^2+K+2}{K-1}\tfrac{1}{K+1}\tfrac{\gamma}{\gamma+1}+\tfrac{1}{\gamma+1}+\tfrac{2}{(K-1)^2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{K+1}\right)^{\gamma} \\
&= -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\
&+\tfrac{1}{\gamma+1}\tfrac{K^2-2K+3}{(K-1)^2}\left(\tfrac{1}{K+1}\right)^{\gamma}-\tfrac{\gamma}{K+1}\left(K^2-K-2\right)\tfrac{1}{K-1}\left(\tfrac{1}{K+1}\right)^{\gamma}
\end{aligned}$$

Differentiating then again with respect to K yields:

$$\frac{\partial^2 D_r(1/2,K,\gamma)}{\partial K^2} = = \frac{2\frac{\left(\frac{1}{2}\right)^{\gamma-1}}{(\gamma+1)(K-1)^3}-\frac{1}{\gamma+1}\frac{\left(\frac{1}{K+1}\right)^{\gamma}}{(K-1)^3(K+1)}\times}{\left(K^3\left(-\gamma^2+\gamma\right)+2K^2\left(2\gamma^2-\gamma\right)+5K\left(-\gamma^2+\gamma\right)+4K+2\left(\gamma-1\right)^2+2\right)} = -$$

Hence

$$\frac{\partial^2 D_r(1/2,K,\gamma)}{\partial K^2} < 0$$

if and only if

$$G(K) \equiv \frac{\left(\frac{K+1}{2}\right)^{\gamma+1}}{\frac{(K-1)^2}{8}\left(K\left(\gamma-\gamma^2\right)+2\gamma^2\right)+\gamma\frac{(K-1)}{2}+\frac{(K+1)}{2}} < 1 \qquad (33)$$

First, we observe that $G(1)=1$. Thus, to establish (33), it is enough to show that G is decreasing for all $K>1$. Letting $x \equiv K-1$ for notational simplicity, $\frac{dG(K)}{dK} < 0$ if, and only if,

$$\frac{\frac{d}{dx}\left(\left(\frac{x}{2}+1\right)^{\gamma+1}\right)}{\left(\frac{x}{2}+1\right)^{\gamma+1}} < \frac{\frac{d}{dx}\left(\gamma x\left(\frac{x}{8}\left(x\left(1-\gamma\right)+1+\gamma\right)+\frac{1}{2}\right)+\frac{x}{2}+1\right)}{\left(\gamma x\left(\frac{x}{8}\left(x\left(1-\gamma\right)+1+\gamma\right)+\frac{1}{2}\right)+\frac{x}{2}+1\right)},$$

or:

$$\frac{\frac{\gamma+1}{2}\left(\frac{x}{2}+1\right)^{\gamma}}{\left(\frac{x}{2}+1\right)^{\gamma+1}} = \frac{\gamma+1}{x+2} < \frac{\frac{1}{2}\gamma+\frac{1}{4}x\gamma+\frac{1}{4}x\gamma^2+\frac{3}{8}x^2\gamma-\frac{3}{8}x^2\gamma^2+\frac{1}{2}}{\frac{x}{2}+\frac{1}{2}x\gamma+\frac{1}{8}x^2\gamma+\frac{1}{8}x^3\gamma+\frac{1}{8}x^2\gamma^2-\frac{1}{8}x^3\gamma^2+\frac{1}{2}}$$

[26] To complete the argument we verify the claimed continuity property of $D_r(K,\gamma)$, as in (29), at $K=1:$

$$\begin{aligned}
&\lim_{K\to 1} D_r(K,\gamma) \\
&= \left(\frac{1}{2}\right)^{\gamma}\frac{1}{\gamma+1}-\lim_{K\to 1}\frac{1}{K-1}\frac{2}{\gamma+1}\left[\left(\frac{1}{K+1}\right)^{\gamma}-\left(\frac{1}{2}\right)^{\gamma}\right] \\
&= \left(\frac{1}{2}\right)^{\gamma}\left(\frac{1}{\gamma+1}\right)-\frac{-\gamma 2^{\gamma}}{(\gamma+1)\,4^{\gamma}} = \left(\frac{1}{2}\right)^{\gamma} = D_r(1,\gamma)
\end{aligned}$$

The above inequality is equivalent to the following one:

$$\left(\gamma+\frac{1}{2}x\gamma+\frac{1}{2}x\gamma^2+\frac{3}{4}x^2\gamma-\frac{3}{4}x^2\gamma^2+1\right)(x+2)>$$
$$(\gamma+1)\left(x+\gamma x+\frac{1}{4}x^2\gamma+\frac{1}{4}x^3\gamma+\frac{1}{4}x^2\gamma^2-\frac{1}{4}x^3\gamma^2+1\right),$$

or

$$\begin{array}{c}\gamma x+\frac{1}{2}x^2\gamma+\frac{1}{2}x^2\gamma^2+\frac{3}{4}x^3\gamma-\frac{3}{4}x^3\gamma^2+x\\ 2\gamma+x\gamma+x\gamma^2+\frac{3}{2}x^2\gamma-\frac{3}{2}x^2\gamma^2+2\end{array}>$$
$$\begin{array}{c}x+\gamma x+\frac{1}{4}x^2\gamma+\frac{1}{4}x^3\gamma+\frac{1}{4}x^2\gamma^2-\frac{1}{4}x^3\gamma^2+1+\\ \gamma x+\gamma^2x+\frac{1}{4}x^2\gamma^2+\frac{1}{4}x^3\gamma^2+\frac{1}{4}x^2\gamma^3-\frac{1}{4}x^3\gamma^3+\gamma\end{array},$$

or

$$\frac{7}{4}x^2\gamma+\frac{1}{2}x^3\gamma+\gamma+1+\frac{1}{4}x^3\gamma^3>\frac{1}{4}x^2\gamma^3+\frac{3}{4}x^3\gamma^2+\frac{3}{2}x^2\gamma^2.$$

That is

$$\frac{x^2\gamma}{2}\left(\frac{7}{2}-\frac{\gamma^2}{2}-3\gamma\right)+\frac{1}{4}x^3\gamma\left(2+\gamma^2-3\gamma\right)+\gamma+1>0.$$

the above inequality being always true if $\gamma<1$, which completes the proof. ■

Proof of Proposition 3: From (20) and (29) we get:

$$D_c(K,\gamma)-D_r(K,\gamma) \tag{34}$$
$$=\left(\frac{1}{2}\right)^{\gamma}\left(\frac{1}{K}\right)^{\gamma-1}-K\left(\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}+\frac{2}{K-1}\frac{1}{\gamma+1}\left(\left(\frac{1}{K+1}\right)^{\gamma}-\left(\frac{1}{2}\right)^{\gamma}\right)$$

As shown in Propositions 1 and 2, when $\gamma<1$ and N is even, the optimal structure both for the ring and the completely connected structures has all components of size $K+1=2$. As we noticed, when $K=1$ the pattern of exposure is identical for the ring and the completely connected structure, hence the value of the above expression equals zero in that case, as can be verified.[27] *CAN WE SAY ANYTHING FOR THE CASE N ODD?? - I WOULD NOT INSIST ON THAT FOR THE MOMENT - AC*

Consider now the case $\gamma>1$, for which $K=N-1$ (i.e. minimal segmentation) is optimal for both structures. Evaluating (34) at this value of K we find:

$$D_c(N-1,\gamma)-D_r(N-1,\gamma)=$$
$$=\left[\left(\frac{1}{N}\right)^{\gamma}-\left(\frac{1}{2}\right)^{\gamma}\right]\left[\frac{2}{N-2}\frac{1}{\gamma+1}-\frac{N-1}{1+\gamma}\right]+\left(\frac{1}{2}\right)^{\gamma}\left[\left(\frac{1}{N-1}\right)^{\gamma-1}-\frac{N-1}{1+\gamma}\right]$$

Since $2\leq(N-1)(N-2)$ for $N\geq3$, we have that for all $N>1+(1+\gamma)^{\frac{1}{\gamma-1}}$ the desired conclusion follows:

$$D_c(1/2,K,\gamma)-D_r(1/2,K,\gamma)<0.$$

[27] Strictly speaking, we can show that that its limit for $K\to1$ equals zero.

IS THIS WHERE THE NEED FOR THE CONDITION N SUFF. LARGE COME FROM?? - OK - AC This completes the proof. ■

Proof of Proposition 4: From (20), we can write:

$$D_c(K,\gamma,\gamma',p) = p\,K\,\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,K\,\left(\frac{1}{2K}\right)^{\gamma'}$$

Hence

$$\frac{\partial D_c}{\partial K}(K,\gamma,\gamma',p) = -p\,(\gamma-1)\left(\frac{1}{2K}\right)^{\gamma} - (1-p)\,(\gamma'-1)\left(\frac{1}{2K}\right)^{\gamma'}. \tag{35}$$

and $\frac{\partial D_c}{\partial K} > 0$ is equivalent to

$$(1-p)\,(1-\gamma')\left(\frac{1}{2K}\right)^{\gamma'} > p\,(\gamma-1)\left(\frac{1}{2K}\right)^{\gamma},$$

or, since $\gamma > 1$ and $\gamma' < 1$,

$$K > \frac{1}{2}\left(\frac{p\,(\gamma-1)}{(1-p)\,(1-\gamma')}\right)^{\frac{1}{\gamma-\gamma'}}.$$

This implies that $D_c(K,\gamma,\gamma',p)$ is minimized at the point

$$\hat{K}(p) = \frac{1}{2}\left(\frac{p\,(\gamma-1)}{(1-p)\,(1-\gamma')}\right)^{\frac{1}{\gamma-\gamma'}}$$

provided this point is admissible, i.e. $\hat{K}(p) \in [1, N-1]$.

Compute next the second derivative of $D_c(.)$:

$$\begin{aligned}
\frac{\partial^2 D_c}{\partial K^2}(K,\gamma,\gamma',p) &= p\,(\gamma-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,(\gamma'-1)\,\frac{\gamma'}{K}\left(\frac{1}{2K}\right)^{\gamma'} \\
&\geq p\,(\gamma-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma} + (1-p)\,(\gamma'-1)\,\frac{\gamma}{K}\left(\frac{1}{2K}\right)^{\gamma'} \\
&= -\frac{\gamma}{K}\frac{\partial D_c}{\partial K}(K,\gamma,\gamma',p)
\end{aligned}$$

Thus $\frac{\partial^2 D_c}{\partial K^2}(1/2, K,\gamma,\gamma',p) > 0$ for all feasible $K < \hat{K}(p)$, i.e. the function $D_c(\cdot)$ is convex in this range.

The optimal degree of segmentation for the completely connected structure is obtained as a solution of problem 18. Denote by $(K_i^*)_{i=1}^C$ a vector of component sizes that solves this optimization problem. We will show that there exists some appropriate range $[p_0, p_1]$ such that if $p \in [p_0, p_1]$, the optimal component sizes are such that $K_i^* = K_j^* = K^*$ for all $i, j = 1, 2, \ldots, C$ and some common K^* with $2 \leq K^* \leq N-2$.

Choose p_0 such that $\hat{K}(p_0) = \frac{N}{2} - 1$. Such a choice is feasible and unique since by A.3 $N > 4$, $\hat{K}(\cdot)$ is increasing in p, $\hat{K}(0) = 0$, and $\hat{K}(p) \to \infty$ as $p \to 1$. Next we show that, for all $p \geq p_0$, whenever $C \geq 2$, the vector $(K_i^*)_{i=1}^C$ solving problem 18 satisfies:

$$\forall i, j = 1, 2, \ldots, C, \quad K_i^* = K_j^* \leq \hat{K}(p) \tag{36}$$

Let K_i^* and K_j^* stand for any two component sizes that are part of the solution to the optimization problem. First note that, since $\hat{K}(p) \geq N/2 - 1$, if $K_i^* > \hat{K}(p)$ then we must have that $K_j^* < \hat{K}(p)$. But such asymmetric arrangement cannot be part of a solution to problem 18 because $D_c(\cdot, \gamma, \gamma', p)$ is increasing at K_i^* and decreasing at K_j^*. Hence a sufficiently small increase of K_j and a decrease of K_i, which keeps $K_i + K_j$ unchanged, is feasible and allows to decrease the expected mass of defaults. The only possibility, therefore, is that $K_j^* \leq \hat{K}(p)$ and $K_j^* \leq \hat{K}(p)$.

To complete the argument and establish (36), suppose that at an optimum we have $K_i^* \neq K_j^*$ for at least two components i, j. Since, as shown in the previous paragraph, neither K_i^* nor K_j^* can exceed $\hat{K}(p)$, both K_i^*, K_j^* lie in the convex part of the function $D_c(\cdot, \gamma, \gamma', p)$. It follows, therefore, that if we replace these two (dissimilar) components with two components of equal size $\frac{1}{2}(K_i^* + K_j^*)$, feasibility is still satisfied and the overall expected mass of defaults is reduced, contradicting that the two heterogenous components of size K_i^*, K_j^* belongs to an optimum configuration.

We have thus shown that, when $p \geq p_0$, if at the optimum we have $C \geq 2$, the unique optimal configuration involves a uniform segmentation in components of common size $K^*(p) \leq \hat{K}(p)$. It remains then to show that at the optimum we indeed have $C \geq 2$. At $p = p_0$ the optimum exhibits two components, $C = 2$, since the optimal component size $\hat{K}(p_0) = N/2-1$ is feasible. Since $\hat{K}(p)$ is increasing and continuous in p and $D_c(K, \gamma, \gamma', p)$ is continuous in K, by continuity there exists some p_1, with $p_0 < p_1 < 1$, such that for all $p \in (p_0, p_1)$ the expected mass of defaults in a structure with two components, both of size $N/2-1$, is still smaller than that in a single component of size N. That is, at the optimum C is still equal to 2.

Since $N/2 - 1 > 1$, this completes the proof that the optimal component size $K^* + 1$ is "intermediate," i.e. satisfies $1 < K^* < N - 1$. ■

Proof of Proposition 5: For the probability distribution of the b shock stated in the claim, the expected mass of firms not directly hit by a b shock who default in a completely connected component of size K when a b shock hits the component is:

$$D_c(K, \gamma, p) = (1-p)K + pK \left(\frac{1}{2K}\right)^{\gamma}. \tag{37}$$

Differentiating the above expression with respect to K yields:

$$\frac{\partial D_c(K, \gamma, p)}{\partial K} = (1-p) - (\gamma - 1)\, p \left(\frac{1}{2K}\right)^{\gamma},$$

which is negative for all K as long as (22) is satisfied. This establishes that the optimal degree of segmentation for the complete structure is minimal, that is obtains at $K = N-1$.

Next, using (29) and (17), noting that $\bar{L} > \frac{1}{H} = K+1$ for all K, we obtain the following expression for the expected mass of defaults in the case of the ring structure:

$$D_r(K,\gamma,p) = (1-p)\left(K - \left(K - \frac{2}{K+1}\right)\frac{K+1}{\bar{L}}\right) + p\left[\left(\frac{K}{\gamma+1} - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}\right] + p\left[\frac{1}{K-1}\frac{1}{2^{\gamma-1}}\frac{1}{\gamma+1}\right]$$

It suffices then to show that the expected mass of defaults is smaller for the ring than for the completely connected structure when $K = N-1$: $D_c(N-1,\gamma,p) > D_r(N-1,\gamma,p)$ or, substituting the above expressions:

$$(1-p)(N-1) + p(N-1)\left(\tfrac{1}{2(N-1)}\right)^{\gamma} > (1-p)\left(N-1-\tfrac{N^2-N-2}{2N-1}\right) + p\left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + p\left[\tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1}\right],$$

which can be rewritten as

$$\left(\tfrac{1-p}{p}\right)\tfrac{N^2-N-2}{2N-1} > \left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + \tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1} - (N-1)\left(\tfrac{1}{2(N-1)}\right)^{\gamma}.$$

Using (22) the above inequality holds for an open interval of values of p if

$$(\gamma-1)\left(\tfrac{1}{2(N-1)}\right)^{\gamma}\tfrac{N^2-N-2}{2N-1} > \left(\tfrac{N-1}{\gamma+1} - \tfrac{2}{N-2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{N}\right)^{\gamma} + \tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{\gamma+1} - (N-1)\left(\tfrac{1}{2(N-1)}\right)^{\gamma}$$

or

$$(\gamma-1)\left(\tfrac{1}{2(N-1)}\right)^{\gamma} 2\left(\tfrac{(N-1)^2+N-3}{2(2(N-1)+1)}1\right) + (N-1)\left(\tfrac{1}{2(N-1)}\right)^{\gamma} - 2\left(\tfrac{(N-1)}{2}\left(\tfrac{1}{\gamma+1}\right) - \tfrac{1}{\gamma+1}\left(\tfrac{1}{N-2}\right)\right)\left(\tfrac{1}{N}\right)^{\gamma} - \left(\tfrac{1}{N-2}\tfrac{1}{2^{\gamma-1}}\tfrac{1}{(\gamma+1)}\right) > 0$$

Noticing that by A.3 and (21) we have $N \geq 5$ and this in turn implies

$$\frac{(N-1)^2+N-3}{4(N-1)+2} \geq \frac{N-1}{4},$$

a sufficient condition for the above inequality to hold is that:

$$\begin{aligned}
&(\gamma-1)\left(\frac{1}{2(N-1)}\right)^{\gamma} 2\left(\frac{N-1}{4}\right) + \frac{2}{\gamma+1}\frac{1}{N-2}\left(\frac{1}{N}\right)^{\gamma} \\
&+(N-1)\left(\frac{1}{2(N-1)}\right)^{\gamma} - \left(\frac{N-1}{\gamma+1}\right)\left(\frac{1}{N-1}\right)^{\gamma} - \left(\frac{1}{N-2}\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) \\
= \;& \frac{N-2}{(N-1)^{\gamma-1}}\left((\gamma-1)\frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma}} - \frac{1}{\gamma+1}\right) + \frac{2}{\gamma+1}\left(\frac{1}{N}\right)^{\gamma} - \left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) > 0.
\end{aligned}$$

Since $\gamma \in (1,2)$, this inequality is in turn satisfied if the following hold:

$$\left[\frac{N-2}{(N-1)^{\gamma-1}} + \left(\frac{1}{N}\right)^{\gamma}\right]\left(\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}\right) - \left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) > 0$$

or

$$(N-1)^{2-\gamma} - \left(\frac{1}{(N-1)^{\gamma-1}} - \frac{1}{N^{\gamma}}\right) > \frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}}$$

which is implied by the inequality

$$(N-1)^{2-\gamma}-\left(\frac{1}{4^{\gamma-1}}-\frac{1}{5^{\gamma}}\right)>\frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}}-\frac{1}{\gamma+1}}$$

that is in turn equivalent to (21). This completes the proof of the proposition. ■

Proof of Proposition 6: Let $\mathbf{S}\equiv(S_i)_{i\in N}$ be a CPE of the network-formation game with completely connected components. Denote by C^S the number and by $\{K_j+1\}_{j=1}^{C^S}$ the sizes of the different components of the CPE network $\Gamma(\mathbf{S})$. Recall that $\hat{K}$ denotes the unique value that minimizes $D_c(K,\gamma,\gamma',p)$ (c.f. the proof of Proposition 4).

The proof has two main steps. We show first that for each component $j=1,..,C^S$ we have $K_j\leq\hat{K}$. That is, all components in the CPE have size smaller or equal than the individually optimal one. Suppose not: $K_j>\hat{K}$ for some j. Choose then a subset $\mathcal{D}$ of the firms belonging to component j whose measure satisfies $K_j-\hat{K}>|\mathcal{D}|>0$. Since the function $D_c(K,\gamma,\gamma',p)$ is increasing in K for $K>\hat{K}$, $D_c(K_q-|\mathcal{D}|,\gamma,\gamma',p)<D_c(K_q,\gamma,\gamma',p)$. This implies that all the firms who are in component j except those in $\mathcal{D}$ could enjoy a lower default probability by severing all their linkages with the firms in $\mathcal{D}$, and this deviation is always feasible. This establishes the desired conclusion.

Next, we show that at a CPE we have $K_{\widetilde{h}}<\hat{K}$ for some $\widetilde{h}\in\left\{1,2,...,C^S\right\}$ and $K_j=\hat{K}$ for all $j\neq\widetilde{h}$. We proceed again by contradiction. Suppose that there are two components of $\Gamma(\mathbf{S})$, j and q, such that $0<K_j<K_q<\hat{K}$. Pick then a subset of firms belonging to the first component: $\mathcal{D}_j\subset K_j$, of measure ε with $0<\varepsilon<\hat{K}-K_q$. Consider the following joint deviation for the firms in $\mathcal{D}_j$ as well as for all those in component q from their strategies in $\mathbf{S}$. Each of the firms in $\mathcal{D}_j$ deletes all its links to the firms in component j (except the links to the other firms in $\mathcal{D}_j$) and, at the same time, forms new links with *all* the firms in component q. Also, all firms in component q form new links with the firms in $\mathcal{D}_j$. Then, every firm in $\mathcal{D}_j$ as well as every firm in component q ends up having a total measure of links equal to $K_q+\varepsilon$. Denote by $\tilde{\Gamma}$ the network induced by these deviations. Then, [*maintaining the property that the deviating agents continue to exchange* $1/2$ *of their own assets with all their neighbors (old and new) in a uniform manner WE MAY EVEN DROP THIS NOW],* we have:

$$D_c(K_q+\varepsilon,\gamma,\gamma',p)<D_c(K_q,\gamma,\gamma',p)<D_c(K_j,\gamma,\gamma',p)$$

Hence all firms involved benefit from the deviation, which contradicts the fact that at a CPE we have $0<K_j<K_q<\hat{K}$. This contradiction establishes the above claim and completes so the proof of the Proposition. ■

Remark 2 *It is immediate to check that the line of argument used above in proving Proposition 6 still applies if the notion of CPE were redefined to limit the size/measure of deviating coalitions to be no higher than some pre-established value* η*, arbitrarily small but positive. For, in both of* *the main steps of the proof, the only condition required is that some positive measure of agents be involved, which is obviously consistent with any positive value for the aforementioned* η.

An additional observation worth making is that the deviations contemplated in the proof of Proposition 6 are "internally consistent" in the following sense. Given that any set of firms of the required composition is set to deviate, this deviation is itself in the interest of any subset of this set that might reconsider the situation. This requirement (which is commonly demanded in the game-theoretic literature for coalition-based notions of equilibrium), is clearly satisfied in our case since, by refusing to follow suit with the deviation, any firm in this subset can only lose.

Proof of Proposition 9: To prove the result, we compute the expected number of bankruptcies associated to the two structures (star and symmetric) for all possible levels of the b shock.

Noting that $\alpha' - \alpha = (1-\theta)^2 (\beta - 1)/\beta$, the exposure matrix for the star structures can be conveniently rewritten as follows:

$$\widetilde{A} = \begin{pmatrix} \alpha' & (1-\alpha')/\beta & (1-\alpha')/\beta & \cdots & (1-\alpha')/\beta \\ 1-\alpha' & \alpha & (\alpha'-\alpha)/(\beta-1) & \cdots & (\alpha'-\alpha)/(\beta-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\alpha' & (\alpha'-\alpha)/(\beta-1) & (\alpha'-\alpha)/(\beta-1) & \cdots & \alpha \end{pmatrix},$$

Recall that $\alpha = 1/2$ while α' is determined, together with θ, by (25) and (26). Its properties are characterized below:

LEMMA 3 *For all* $\beta > 2$*, the solution of (25) and (26) is unique and given by continuous, monotonically increasing functions* $\theta(\beta)$ *and* $\alpha'(\beta)$*, such that*

$$5/9 \leq \alpha'(\beta) < 2 - \sqrt[2]{2} \tag{38}$$

Proof of Lemma 3:

It can be easily verified that for all $\beta > 2$ there is only one admissible (i.e., lying between 0 and 1) solution of (25), given by

$$\frac{2 + \sqrt{2\beta^2 - 2\beta}}{2\beta + 2}.$$

This expression defines the function $\theta(\beta)$, which is increasing if and only if the following inequality is satisfied:

$$\frac{(4\beta - 2)}{2\sqrt{2\beta^2 - 2\beta}} (2\beta + 2) > \left(4 + 2\sqrt{2\beta^2 - 2\beta}\right),$$

which is equivalent to

$$2\beta^2 + \beta - 1 > 2\sqrt{2\beta^2 - 2\beta} + 2\beta^2 - 2\beta$$

or

$$9\beta^2 - 6\beta + 1 > 8\beta^2 - 8\beta$$

always satisfied for $\beta > 2$. The minimal value of θ in this range is then $\theta(2) = 2/3$, while the maximum is $\lim_{\beta\to\infty} \theta(\beta) = 1/\sqrt[2]{2}$.

Also, $\alpha'(\beta)$ is also increasing in β

$$\frac{d\alpha'}{d\beta} = 2(2\theta - 1)\frac{d\theta}{d\beta}.$$

Hence its minimum value is $\alpha'(\beta) = 5/9$ and its maximum is $2 - \sqrt[2]{2}$. $\square$

Denote by $G_{star}(L)$ and $G_{sym}(L)$ the functions that specify the total size of all[28] the firms who default, resp. for the star and the symmetric structure, as a function of the magnitude L of the b shock.

*L*et us begin by determining the total size of firms defaulting if the shock hits a *small firm* and the structure of the component to which it belongs is a star. For any given value of L, it is given by the following function[29] :

$$G^s_{star}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} = 2 \\ 1 & \text{for } 2 < L \leq \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ \beta \text{ if } \frac{\beta-1}{\alpha'-1/2} \leq \frac{\beta^2}{1-\alpha'} & \\ 1+\beta \text{ otherwise} & \text{for } \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} < L \leq \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ 2\beta & \text{for } L > \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \end{cases}$$

Instead, if the structure is still a star but the shock hits a *large firm* (i.e. the hub), the expected number of defaults is given by the following function:

$$G^\ell_{star}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{1-\alpha'} \\ \beta & \text{for } \frac{1}{1-\alpha'} < L \leq \frac{\beta}{\alpha'} \\ 2\beta & \text{for } L > \frac{1}{1-\alpha'} \end{cases},$$

since the upper and lower bounds on α' established in the previous Lemma imply that $\beta/\alpha' > 1/(1-\alpha')$. *Ex ante*, a shock hitting a component of the network has the same probability of striking a large firm or a small firm. Hence, $G_{star}(L) = \left(G^s_{star}(L) + G^\ell_{star}(L)\right)/2$, or:

[28]Note that, in contrast to the functions $g_v(L;K)$, $v = c, r$, introduced in Section 2.2, the functions $G_{star}(.)$, $G_{sym}(.)$ describe the number of *all firms* who default, that is including the firm that is directly hit by the shock L. Since the degree of risk externalization, as we noticed, may now differ across different structures, so does the probability that a firm directly hit by a shock defaults.

[29]Recall that a small firm i defaults when a shock hits a firm j if and only if $\tilde{a}_{ij}L > 1$, while a large firm k defaults if and only if $\tilde{a}_{kj}L > \beta$.

$$
G_{star}(L) = \begin{cases} 0 & \text{for } L \leq 2 \\ \frac{1}{2} & \text{for } 2 < L \leq \frac{1}{1-\alpha'} \\ \frac{1}{2} + \frac{1}{2}\beta & \text{for } \frac{1}{1-\alpha'} < L \leq \frac{\beta}{\alpha'} \\ \frac{1}{2} + \frac{1}{2}2\beta & \text{for } \frac{\beta}{\alpha'} < L \leq \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ \begin{array}{c} \frac{\beta}{2} + \beta \text{ if } \frac{\beta-1}{\alpha'-1/2} \leq \frac{\beta^2}{1-\alpha'} \\ \frac{1+\beta}{2} + \beta \text{ otherwise} \end{array} & \text{for } \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} < L \leq \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \\ 2\beta & \text{for } L > \max\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\} \end{cases}, \tag{39}
$$

since again it can be verified, given the previous lemma, that $2 < \frac{1}{1-\alpha'} < \frac{\beta}{\alpha'} < \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\}$.

Consider next the case where the structure is symmetric, with two completely connected components, the first one with the two large firms, the second one with the 2β small firms. In this case, since there is no asymmetry within each component, every firm retains a fraction exactly equal to $\alpha = 1/2$ of claims to the returns of its own project. The off diagonal terms of the exposure matrix are then equal to $1/2$ for the component with the two large firms and $1/[2(2\beta - 1)]$ for the second one, with 2β firms. Since the shock reaches with equal probability each of the two components, the expected number of defaults is given by the following function:

$$
G_{sym}(L) = \begin{cases} 0 & \text{for } L \leq 2 \\ \frac{1}{2} & \text{for } 2 < L \leq 2\beta \\ \frac{1}{2} + \frac{1}{2}2\beta & \text{for } 2\beta < L \leq 2(2\beta - 1) \\ 2\beta & \text{for } L > 2(2\beta - 1) \end{cases} \tag{40}
$$

since it can be easily verified that $2 < 2\beta < 2(2\beta - 1)$.

A straightforward comparison of the functions $G_{star}(L)$ and $G_{sym}(L)$ given in (39) and (40), noting that $\frac{\beta}{\alpha'} < 2\beta$ and $\frac{\beta}{\alpha'} < 2(2\beta - 1) < \min\left\{\frac{\beta-1}{\alpha'-1/2}, \frac{\beta^2}{1-\alpha'}\right\}$, yields then the claim in the proposition. ■

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